THE EIGHTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE eighteenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 27–28, 1911, the programme occupying two sessions on each day. The total attendance numbered about seventy-five, including the following sixty-four members:

Dr. C. S. Atchison, Mr. H. Bateman, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor B. H. Camp, Professor W. M. Carruth, Dr. A. S. Chessin, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Professor J. C. Fields, President H. B. Fine, Professor W. B. Fite, Mr. Meyer Gaba, Professor A. S. Gale, Professor O. E. Glenn, Mr. G. H. Graves, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Professor C. R. MacInnes, Professor J. H. Maclagan-Wedderburn, Professor James Maclay, Professor G. A. Miller, Professor C. L. E. Moore, Professor Richard Morris, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Mr. L. P. Siceloff, Mr. C. G. Simpson, Professor P. F. Smith, Mr. W. M. Smith, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Dr. W. A. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

President Fine occupied the chair, being relieved at the Wednesday afternoon session by Ex-President White. The Council announced the election of the following new members of the Society: Professor Ida Barney, Rollins College; Professor Louis Brand, University of Cincinnati; Professor C. W. Cobb, Amherst College; Professor J. C. Fitterer, University of Wyoming; Mr. G. H. Graves, Columbia University; Dr.

Solomon Lefschetz, University of Nebraska; Mr. G. H. Light, Purdue University; Mr. E. S. Palmer, Rutgers College; Professor E. R. Smith, Pennsylvania State College. Eight applications for membership in the Society were received.

The annual dinner on Wednesday evening was a very pleasant occasion, forty-two members being present. These informal gatherings have long been recognized as one of the most attractive features of the meetings.

The reports of the Treasurer, Auditing Committee, and Librarian have appeared in the Annual Register. The membership of the Society has risen to 668, including 62 life members. The total attendance of members at meetings during the year just passed was 423; the number of papers read was 217. At the annual election 197 votes were cast. The Society's library now contains 3,871 volumes, beside some 500 unbound dissertations. The Treasurer's report shows a balance of $8,723.89, but the increase is offset by outstanding bills amounting to about $550. The income from sales of the Society's publications during the year was $1,513.66. The life membership fund now amounts to $4,137.17.

With the great growth of the Society's activities it will soon become necessary to make some provision for clerical services, the capacity of the several offices being already overstrained in this direction. The Society has outgrown the stage in which its officers could attend personally to all the details of administration.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professor H. F. BLICHFELDT,
Professor HENRY TABER.

Secretary, Professor F. N. COLE.

Treasurer, Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1914.

Professor A. B. COBLE,
Professor E. W. DAVIS,
Professor OSWALD VEBLEN,
Professor E. B. WILSON.

The following papers were read at this meeting:

(1) Mr. W. M. SMITH: "A characterization of isogonal and equitangential trajectories."

(2) Professor C. L. E. MOORE: "Surfaces in hyperspace which have a tangent line with three-point contact passing through each point."

(3) Dr. J. E. ROWE: "How to find a set of invariants for any rational curve of odd order."

(4) Dr. J. E. ROWE: "A covariant point of the R^4, and a special canonical form."

(5) Dr. R. L. MOORE: "On sufficient conditions that an integral equation of the second kind shall have a continuous solution."

(6) Professor E. B. WILSON: "Some mathematical aspects of relativity."

(7) Professor EDWARD KASNER: "Families of surfaces related to an arbitrary deformation of space."

(8) Dr. H. B. PHILLIPS and Professor C. L. E. MOORE: "Algebra of plane projective geometry."

(9) Professor ANNA L. VAN BENSCHOTEN: "Products of quadric inversions and linear transformations in space" (preliminary report).

(10) Professor ARTHUR RANUM: "N-dimensional spreads generated by ∞^1 flats."

(11) Professor O. E. GLENN: "Generalizations of a theorem on reducible quantics, due to Eisenstein."

(12) Professor F. R. SHARPE: "Finite groups of birational transformations in the plane."

(13) Professor JOHN EIESLAND: "On a flat spread-sphere geometry in an odd dimensional n-space."

(14) Professor C. N. MOORE: "The summability of the double Fourier series, with applications."

(15) Professor S. E. SLOCUM: "A general formula for torsional deflection."

(16) Professor G. A. MILLER: "Groups which contain a given number of operators whose orders are powers of the same prime."

(17) Professor R. G. D. RICHARDSON: "Theorems of oscillation for three self-adjoint linear differential equations of the second order with three parameters."

(18) Professor L. A. HOWLAND: "Points of undulation of algebraic plane curves."

(19) Professor C. N. HASKINS: "Note on certain selective integrals."

(20) Professor L. P. EISENHART: "Ruled surfaces with isotropic generators."

(21) Professor J. W. YOUNG: "On algebras defined by groups of transformations."

(22) Professor J. W. YOUNG: "A generalization to 3-space and to n-space of the inversion geometry in a plane" (preliminary communication).

(23) Dr. L. L. SILVERMAN: "On absolute or unconditional summability."

(24) Dr. L. L. SILVERMAN: "Tests for Cesàro summability."

(25) Professor W. B. FITE: "Note on a collineation group in n variables."

(26) Professor L. P. EISENHART: "Congruences of minimal lines in a four-space."

In the absence of the authors the papers of Dr. Moore, Dr. Sharpe, Professor Eiesland, Professor C. N. Moore, Professor Slocum, and Dr. Silverman were read by title. Abstracts of the papers follow below.

1. Cesàro has shown that in any complete isogonal system the locus of the centers of curvature of the single infinity of trajectories passing through the same point is a straight line. Similarly, Scheffers has proved that in any complete equitangential system those trajectories tangent to the same straight line have their centers of curvature upon a straight line. The former is here referred to as the Cesàro property and the latter as the Scheffers property.

In any doubly infinite system of plane curves any one of the single infinity of trajectories through a point of the plane has a certain curvature γ. Let N denote the rate of variation of γ along the normal, the direction of the consecutive trajectory remaining the same, and let γ_1, N_1 denote the corresponding quantities for that trajectory which is orthogonal to the first. Mr. Smith shows that a doubly infinite system of plane curves whose single infinity of trajectories through any point possess the property expressed by

$$N + N_1 = \gamma^2 + \gamma_1^2$$

together with the Cesàro property is a complete isogonal system.

If it possesses this property together with the Scheffers property, it is a complete equitangential system.

2. In this paper Professor Moore shows that the coordinates of a surface in hyperspace, having through each point a tangent line with three-point contact, must satisfy a partial differential equation of parabolic type. If the coordinates satisfy such an equation, the surface will be of the kind mentioned. Similar results are obtained for other spreads in hyperspace.

3. An algebraically complete system of invariants of the rational plane quartic has been found by Dr. Rowe in a previous paper presented to the Society September 6, 1910. (See *Transactions*, July, 1911.) This enables us to find a set of invariants for any rational plane curve of odd order, which is called R^{2n+1}. Every R^{2n+1} has a covariant rational line quartic ρ^4, whose parametric equations may be written down, and from these the four fundamental invariants for each ρ^4 may be calculated. This work has been carried out in detail for the R^5.

4. In Dr. Rowe's second paper a covariant point of the R^4, not on the R^4, is found which possesses the following properties: (1) tangents to R^4 from this point are apolar to the six flex parameters and to the q's each taken three times; (2) the line cutting out the q's on R^4 and the line cutting them out of Stahl's conic N meet in this point; (3) the flex lines of two cubic osculants whose parameters are given by a binary quadratic meet in this point. The coordinates of the point are of degree four in the coefficients of the parametric expressions of the R^4.

In the second part of the paper a special canonical form of the R^4 is discussed which enables us to express in terms of the four fundamental invariants conditions on the R^4 which are rather difficult to attack directly. For instance, the condition that two of a set of covariant parameters, cut out of the R^4 by a covariant curve, form a node is easily handled by means of this canonical form. The flecnode and tacnode conditions occur as special cases of this.

5. Making use of a theorem on which he reported at the April, 1911, meeting of the Society (cf. BULLETIN, volume 17, No. 10, page 513), Dr. Moore proposes to show that

if $f(x)$ is continuous throughout the interval

$$I: \quad a \leqq x \leqq b$$

and $k(x, y)$ is a limited function of x and y in the square

$$S: \quad a \leqq x \leqq b, \quad a \leqq y \leqq b,$$

and if the Fredholm determinant D is not 0, and finally if on each parallel to the axis of x or of y the points of discontinuity, if any, of $k(x, y)$ which lie in S form a point set of linear measure 0, then the integral equation of the second kind

$$u(x) = f(x) + \int_a^b k(x, y)u(y)dy$$

has a solution $u(x)$ which is continuous throughout I.

6. Professor Wilson comments on some points of a mathematical nature in a memoir which he and Professor G. N. Lewis, of the Massachusetts Institute of Technology, are writing on the principle of relativity. The occurrence of a parabolic non-euclidean geometry and of a corresponding non-circular vector analysis, and the possibility of the use of potentials in four dimensions are the two chief subjects treated. The former was in part set forth and in part implied in Minkowski's work; the latter has been suggested by Sommerfeld.

7. Professor Kasner discusses the existence and determination of surfaces which are affected conformally, or more specially isometrically, by a given deformation of space. Unless the transformation of space is conformal there cannot be more than six simply infinite families of surfaces affected conformally; and unless the deformation is a displacement there cannot be more than two families affected isometrically.

8. From the assumption that the sum of two points obeys the ordinary laws of algebraic addition, Dr. Phillips and Professor Moore show that there are just two kinds of addition possible, depending upon the interpretation given to a number times a point. If λA is coincident in position with A, then the sum of two points is the harmonic of a fixed line f with respect to the two points. If λA differs in position from A, then the

sum of two points becomes similar to the ordinary vector sum. This second kind of addition leads to a sort of vector analysis in which vector points appear as well as vector lines. Both additions have two fundamental or exceptional elements consisting of a line and a point.

Distance is defined as a scalar quantity uniquely determined by two points and such that distances AB along a line are proportional to the corresponding vectors AB. On these assumptions is built a theory of distance. As a consequence of the definition, to each line of the plane is assigned a definite positive direction. Angle is taken as the exact dual of distance. In the geometry in which distance and angle are defined as above, the locus of points equidistant from A is a line. Also it is found that any three parts of a triangle determine it.

It is assumed for the area of a triangle that, holding the vertex fixed and moving the base of constant length along a line, it will remain constant. It is then found that the area must be a multiple of the sum of the sides.

The product AB of two unit points is defined as the segment connecting the two points. The product of three unit points is defined as the area of the triangle having the three points as vertices. Similar definitions are given for the products of lines.

9. In this report Miss Van Benschoten presents some types of (2, 2), (2, 3), (3, 3), and (3, 4) birational transformations which are obtained as products of quadric inversions and linear transformations in space of three dimensions. It is proved that the (3, 3) transformation in which for each variable its reciprocal is substituted can be resolved in $\frac{1}{2}n(n+1)$ different ways into the product of $n-1$ inversions and a linear transformation in space of n dimensions.

10. This paper is a continuation of the one read by Professor Ranum at the summer meeting of the Society. Taking the ith tangent spread of the jth focal spread of the kth tangent spread of a given spread S, and letting i, j, k vary, we obtain a finite system of spreads determined by S. A number of properties of this system are found. Among the theorems proved is the following:

If S and its 1st, $\cdots$, $(i-1)$-th tangent spreads are all of rank r and its ith tangent spread S_1 of rank $s_1 < r$, while its

1st focal spread S_2 is of rank $s_2 < r$ $(s_1 + s_2 \geqq r)$, and if either T_1, the $(i+1)$-th focal spread of S_1, or T_2, the $(i+1)$-th tangent spread of S_2, is of rank s_1+s_2-r, then the other is also of the same rank and T_1 is the $(i+1)$-th focal spread of T_2. This remarkable case first occurs in 9-dimensional space.

11. Eisenstein's theorem may be stated as follows: A necessary condition that a binary form a_x^m with integral coefficients a_0, a_1, $\cdots$, a_m, all of whose coefficients except a_0 are divisible by a prime number q, should be reducible in the absolute field is that

$$a_m \equiv 0 \pmod{q^2}. \tag{1}$$

Professor Glenn proves corresponding theorems for p-ary forms. He finds that the criteria imposed by these theorems become sharper as p increases, due to the fact that the number of conditions corresponding to (1) is, in the general case, an increasing function of p. A class of theorems on the existence of a factor of a p-ary form whose order lies within limits is established. A new method of resolving a reducible p-ary form is developed. It is proved for a linearly factorable p-ary form that, if a certain related binary form is linearly factorable in a given domain, the p-ary form is reducible in the same domain, though not necessarily completely (linearly) reducible unless the factors of the binary form are all simple. Necessary arithmetical conditions that the p-ary form be completely reducible in that domain are then developed. These take the form of higher order congruences in one unknown. The last section of the paper is devoted to the theory of these congruences.

12. Noether* has shown that a single plane can be depicted upon a double plane so that the double curve is a sextic with two coincident triple points. S. Kantor† has proved that the finite group of birational transformations having 8 fundamental points in the single plane is isomorphic with the group of linear transformations which leave the sextic in the double plane invariant. Wiman‡ used the Grassmann depiction of the single plane upon a cubic surface and thence upon the double plane.

* *Erlangen Berichte*, 1878.
† *Acta Math.*, vol. 19.
‡ *Math. Annalen*, vol. 48.

He also completely enumerated the linear transformations of the sextic. In Professor Sharpe's paper a method is developed for determining the transformation of the single plane which corresponds to a given linear transformation of the double plane. It is known that there are 120 tritangent conics to the sextic which are also tangent to the sextic at the triple point. It is shown that each conic leads to a partial determination of the cubic surface and that the cubic surface can then be completely determined in 28 ways. In a special case the 120 conics and the transformations of the single plane are completely determined.

13. Professor Eiesland's paper is in abstract as follows: A generalization of Lie's sphere geometry becomes possible if, instead of using the straight line as space element, we introduce a flat spread (hyperplane) of $\frac{1}{2}(n-1)$ dimensions. In fact, since to spheres that touch must correspond intersecting flats, the condition for such intersection must be unique; moreover, $\frac{1}{2}(n-1)$ must be an integer, hence, a flat spread-sphere geometry is possible only in a space of an odd number of dimensions. It is also necessary that the number of essential parameters of the flat shall be equal to $n+1$, which is the number of parameters of the sphere.

It has been shown that for an odd space Lie's contact transformation can be generalized and that this new transformation converts the ∞^n spheres in S_{n-1} (n even)

$$\sum_{1}^{n-1}(x_i-\xi_i)^2+\xi_n^2=0$$

into the ∞^n flat spreads

$$x_i=ay_i+b_i,\quad z=\sum_{1}^{n-2}c_iy_i+d\qquad(i=1,2,\cdots,\tfrac{1}{2}(n-2))$$

in a space

$$\bar{S}_{n-1}(x_1,\,y_1,\,x_2,\,y_2,\,\cdots,\,x_{\frac{1}{2}(n-2)},\,y_{\frac{1}{2}(n-2)})$$

with the following relation between the ξ_i's and the parameters a, b_i, c_i, d:

$$\begin{aligned}&\xi_1=-\tfrac{1}{2}(b_1+c_1),\quad \xi_2=\tfrac{1}{2}i(b_1-c_1),\cdots,\\ (1)\qquad &\xi_{n-2}=\tfrac{1}{2}i(b_{\frac{1}{2}(n-2)}-c_{\frac{1}{2}(n-2)}),\\ &\xi_{n-1}=\tfrac{1}{2}(a-d),\quad \xi_n=-\tfrac{1}{2}i(a+d).\end{aligned}$$

The condition that two flats shall intersect is

$$(\Sigma c_i' - c_i)(b_i' - b_i) + (d' - d)(a - a') = 0,$$

which, on introducing the parameters ξ_i from (1), becomes the condition that two spheres shall touch.

The transformation in question is

$$X_1 = \frac{(y_1 - p_1)(\Sigma y_i q_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)},$$

$$X_2 = i\frac{(y_1 + p_1)(z - \Sigma y_i q_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

$$X_{n-1} = \frac{\Sigma y_i q_i - z}{1 - \Sigma p_i y_i},$$

$$P_1 = \frac{y_1 - p_1}{1 + \Sigma p_i y_i}, \quad P_2 = \frac{-i(y_1 + p_1)}{1 + \Sigma p_i y_i}, \cdots,$$

$$P_{n-2} = \frac{-i(y_{\frac{1}{2}(n-2)} + p_{\frac{1}{2}(n-2)})}{1 + \Sigma p_i y_i}.$$

Furthermore, this transformation converts the lines of curvature on an $n - 2$ dimensional spread in S_{n-1} into curves on the corresponding surface which are the analogues of asymptotic curves on a surface in 3-space. These lines have therefore been called asymptotic lines. Through any point on the surface pass $n - 2$ such lines and no more.

Finally *all* the contact transformations of this kind are considered. It is found that they are obtained by superposing the transformation (2) on the transformations of the group of contact transformations in S_{n-1} which change spheres into spheres, and also on the duality transformation. To the before mentioned group in S_{n-1} corresponds in $\overline{S}_{n-1}$ a group which leaves invariant the ∞^n flats, and transforms asymptotic lines into asymptotic line. This group is a subgroup of a group which is similar to the group of contact transformations discussed by Lie in the second volume of his large work on transformation group, chapter 25, where he shows that the transformations of the group leave invariant a hypersurface

$$z = a + \Sigma b_i x_i + \sum_{ik}^{1,\ldots,n-1} c_{ik} x_i x_k.$$

14. Fejér's idea of studying the summability of the Fourier development of an arbitrary function can be extended to the development in a double Fourier series of a function of two variables. In Professor Moore's paper this extension is carried out, and it is shown that the Fourier development of a function of two variables, $f(x, y)$, continuous in the region $(-\pi \leqq x \leqq \pi, -\pi \leqq y \leqq \pi)$, except for a finite number of lines of discontinuity consisting of parallels to the axes or of curves that cut such parallels in a finite number of points, will be summable to the value of the function at every point in the interior of the region at which the function is continuous, provided the discontinuities of the function are of such a nature that the double integrals in the coefficients of the Fourier's series converge. Along the lines of discontinuity the series will be summable at points where the function approaches a definite value as we approach the point from either side of the line, and its value will be half way between the limiting values provided the line of discontinuity has a tangent at the point. The definition of summability adopted is analogous to that adopted by Cesàro for ordinary series. The detailed formulas for positive integral orders of summability have been given in the abstract of a paper previously presented to the Society*; the present paper considers only summability of order one.

Furthermore, the present paper shows how the results stated above, together with the results of the previous paper just referred to, can be applied to the solution of certain problems in the flow of heat.

15. In a previous article (*Journal of the Franklin Institute*, April, 1911) Professor Slocum has deduced a general formula for shearing deflection, and has also given a new and simplified proof of the Fraenkel formula for flexural deflection. In the present paper this work on deflection is completed by the derivation of a general formula for torsional deflection.

* BULLETIN, vol. 17 (1911), p. 530. There is an error in condition (*b*) of the theorem stated in this abstract. The two first equations of this condition should be replaced by the equations

$$\lim_{m=\infty} m^k \sum_{j=\mu_m}^{j=\nu_m} j^k \, | f_{m,j}(\alpha, \beta) | = 0, \quad \lim_{n=\infty} n^k \sum_{i=\mu_n}^{i=\nu_n} i^k \, | f_{i,n}(\alpha, \beta) | = 0,$$

where μ_m and ν_m vary with m, and μ_n and ν_n vary with n in any manner whatsoever.

Let M = torsional moment, or torque, acting in a plane perpendicular to the axis of the piece,
ϑ = torsional deflection, or angle of twist, produced by M,
q = shearing stress per unit area of cross section,
q_x, q_y = rectangular components of q,
φ = angular, or shearing, distortion corresponding to the stress q,
G = modulus of rigidity, or shear modulus,
W = internal work of deformation,
A = area of cross section,
l = length of member.

The internal work of deformation is first found in the form

$$W = \frac{1}{2G}\int_0^l \int_0^A q^2 dA dx.$$

For a prismatic piece of constant cross section this becomes

$$W = \frac{l}{2G}\int_0^A q^2 dA.$$

The following theorem is then proved:

If ϑ denotes the torsional deflection produced by a torque M, and W the total work of deformation arising from a torque $M + K$, then

$$\vartheta = \left[\frac{dW}{dK}\right]_{k=0}.$$

Inserting in this expression the value previously found for W in terms of the shearing stress q, the required formula for torsional deflection is obtained, namely

$$\vartheta = \frac{l}{GM}\int_0^A q^2 dA.$$

To illustrate the application of this formula, it is then used for calculating the torsional deflection of prismatic pieces the cross sections of which are respectively circular, elliptical, triangular, rectangular, and square.

16. If p^m is the highest power of the prime number p which divides the order of a group G, the number of the operators of G whose orders are powers of p is a multiple of p^m, according

to a theorem due to Frobenius. It is, however, not possible to construct a group in which this number is an arbitrary multiple of p^m. In fact, it is at least p^{m+1} whenever it exceeds p^m, and it is at least $2p^{m+1} - p^m$ whenever it exceeds p^{m+1}.

Professor Miller proves, as a special case of a general theorem, that a necessary and sufficient condition that any group contains exactly $p+1$ Sylow subgroups of order p^m is that it contains exactly p^{m+1} operators whose orders are powers of p. He also established the following theorems: If every possible pair of Sylow subgroups of order p^m in a given group has p^{m-1} common operators, then all of these subgroups have the *same* p^{m-1} common operators. If any group contains less than $(p+1)^2$ Sylow subgroups of order p^m, then all the operators which are common to some pair of these subgroups must be common to all of them. If G contains exactly $2p^{m+1}p^m$ operators whose orders are powers of p, it contains also exactly $2p+1$ subgroups of order p^m and all of these involve the same p^{m-1} operators.

17. At the April meeting of the Society, Professor Richardson gave necessary conditions and sufficient conditions for the existence of solutions of the Klein oscillation problem for two equations with two parameters. In the present paper he discusses the problem for three equations with three parameters,

$$[p_i(x_i)u_i'(x_i)]' + q_i(x_i)u_i(x_i) + [\lambda A_{i1}(x_i) + \mu A_{i2}(x_i) + \nu A_{i3}(x_i)]u_i(x_i) = 0, \quad p_i(x_i) > 0 \quad (i = 1, 2, 3).$$

In order that there exist solutions $u_1(x_1)$, $u_2(x_2)$, $u_3(x_3)$ which satisfy the boundary conditions

$$u_1(a_1) = u_1(b_1) = 0, \quad u_2(a_2) = u_2(b_2) = 0,$$

$$u_3(a_3) = u_3(b_3) = 0,$$

and oscillate l, m, n times inspectively, it is necessary that parameter values $\lambda^{(1)}$, $\mu^{(1)}$, $\nu^{(1)}$; $\lambda^{(2)}$, $\mu^{(2)}$, $\nu^{(2)}$; $\lambda^{(3)}$, $\mu^{(3)}$, $\nu^{(3)}$ exist such that $\lambda^{(i)}A_{i1} + \mu^{(i)}A_{i2} + \nu^{(i)}A_{i3} \leqq 0$ for at least one value of the variable x_i in the interval a_ib_i and that $\lambda^{(i)}A_{j1} + \mu^{(i)}A_{j2} + \nu^{(i)}A_{j3} \geqq 0$, $\lambda^{(i)}A_{k1} + \mu^{(i)}A_{k2} + \nu^{(i)}A_{k3} = 0$ for all values of the variables x_j, x_k in the intervals a_jb_j, a_kb_k respectively $(i = 1, 2, 3; j = 1, 2, 3; k = 1, 2, 3; i \neq j \neq k)$. The suf-

ficient conditions for $l \geqq l_1$, $n \geqq n_1$, $m \geqq m_1$ ($l_1 = 0$ when $q = 0$) are the same when the sign $\leqq$ is replaced by the sign $<$.

Another sufficient condition (in this case for all values of l, m, n) is that the determinant of the A's be different from zero for all values of the variables. The solution is then unique.

This paper is to appear in the *Mathematische Annalen.*

18. Points of undulation or, in general, points of an algebraic plane curve at which the tangent has contact of order higher than 2 receive little attention in elementary discussions of the theory of such curves. Or, in other words, although the behavior of the Hessian in multiple points of the curve is treated as fully as may be by elementary methods, the behavior of the curve in multiple points of the Hessian, the consequence of contact between curve and Hessian, or, more generally, of multiplicity of intersection of curve and Hessian greater than $3k(k-1)$ in an ordinary k-fold point are not treated at all.

The object of Professor Howland's paper is to show that these questions can be discussed with the same degree of completeness by elementary methods as can the theory of ordinary inflexions. Among other results the formula $\Sigma(\sigma - 1) = 3n(n-2)$, which applies to curves without multiple points, is extended to curves having ordinary multiple points, in the form $\Sigma(\sigma - 1) = 3n(n-1) - \Sigma 3k(k-1)$, where σ is the order of contact of curve and tangent at each point and Σ is extended to all points of the curve.

19. Landau* and Lebesgue† have recently discussed certain formulas by means of which a given continuous function can be expressed to a preassigned degree of approximation by means of an analytic function. The latter, in particular, has given an exceedingly general theorem by means of which such approximate representations can be constructed. The note of Professor Haskins discusses a similar theorem, somewhat less general than that of Lebesgue, which, however, throws some light on the mechanism of the process of representation of a function by means of a definite integral. The theorem is: If in the interval $a \leqq x \leqq b$ $f(x)$ is continuous,

* *Rendiconti del Circolo Matematico di Palermo*, vol. 25 (1908), p. 337.

† Ibid., vol. 26 (1908), p. 325.

and $\varphi(x)$ is continuous and positive and attains its maximum value M only once in the interval, namely at the point $x=x_0$, then

$$\lim_{n=+\infty} \frac{\int_a^b f(x)\phi(x)^n dx}{\int_a^b \phi(x)^n dx} = f(x_0).$$

The theorem is closely related to a theorem of Stieltjes,* and the proof is based on methods similar to that used in Graeffe's method for determining the roots of an algebraic equation with numerical coefficients.

20. Ruled surfaces with isotropic generators have been considered by Monge, J. A. Serret, Lie, and others. Professor Eisenhart uses for the determination of such a surface the curve in which it is cut by a plane and the directions of the lines which are the projections on the plane of the generators of the surface. In this way a ruled surface of this type is determined by a set of lineal elements, in a plane, depending upon one parameter. The envelope of the direction of the lineal elements is a generalization of oblique evolutes, or evolutoids. Such a surface is determined also by a curve in space, in the sense that it is one of the two sheets of the envelope of a certain family of spheres whose centers are on the space curve. The equations of the two sheets assume a symmetrical form when the space curve is defined in two ways in terms of the two "normal parameters," as defined by the author in the September number of the *Annals*. The relations between the space curve and the set of lineal elements on the plane are very interesting, and lead to a significant transformation of sets of plane curves. Of particular interest is the case in which the spheres are of constant radius, so that the Gaussian curvature of the surface is a positive constant, and thus the ruled surface is a deform of a sphere in which one set of numerical generators remain right lines.

21. In his first paper Professor Young uses the essential ideas of von Staudt's algebra of points on a line to develop the general notion of an algebra associated with certain groups

* Correspondance d'Hermite et de Stieltjes, vol. 2 (1905), p. 185.

of transformations. He considers first a group $\mathfrak{G}$ of transformations which operates in a simply transitive way on the elements $a, b, c, \cdots$ of a certain class Σ. By choosing any particular element of Σ, say i, to be the so-called identical element, there exists in $\mathfrak{G}$ a unique transformation G_a, such that $G_a(i) = a$. An operation $\circ$ acting on pairs of elements of $\mathfrak{G}$ is then defined by the relation $a \circ b = G_a(b)$. The operation $\circ$ is then associative; it is commutative if $\mathfrak{G}$ is commutative; the element i satisfies the relations $a \circ i = i \circ a = a$ for every a of Σ; and corresponding to every a in Σ there exist in Σ two unique elements a', a'' such that $a \circ a' = i$ and $a'' \circ a = i$. The group $\mathfrak{G}$ is then represented by the equations $x' = a \circ x$, the individual transformations of $\mathfrak{G}$ being obtained by allowing a to vary over Σ. The equations $x' = x \circ a$ define a group $\overline{\mathfrak{G}}$, simply isomorphic with $\mathfrak{G}$. $\overline{\mathfrak{G}}$, which is called the adjoint group of $\mathfrak{G}$, is identical with $\mathfrak{G}$ only when $\mathfrak{G}$ is commutative. The author considers next two groups, say $\mathfrak{A}$ and $\mathfrak{M}$, acting in a simply transitive way on two classes Σ_1 and Σ_2 respectively. The operations defined as above by $\mathfrak{A}$ and $\mathfrak{M}$ are for convenience called "addition" and "multiplication" and denoted by $+$ and $\cdot$ respectively; the identical element with respect to $+$ is denoted by 0 and the identical element with respect to $\cdot$ by 1. If Σ_1, Σ_2 have a non-empty class Σ in common, there will be relations between addition and multiplication determined by the relations between the groups $\mathfrak{A}$ and $\mathfrak{M}$. If, in particular, $\mathfrak{M}$ transforms $\mathfrak{A}$ into itself, there is obtained a generalized law of distributivity, expressed by the following relation: $a \cdot (b+c) = a \cdot b + (a \cdot 0)' + a \cdot c$, where the notation m' is used for the element such that $m' + m = 0$. The ordinary form of distributivity is obtained if $a \cdot 0 = 0$ for all values of a; i. e., if every transformation of $\mathfrak{M}$ transforms 0 into itself (0 is then not an element of Σ_2; it is exceptional for $\mathfrak{M}$). The distributive relation just given implies the commutativity neither of addition nor of multiplication. The condition $\mathfrak{M}\mathfrak{A}\mathfrak{M}^{-1} = \mathfrak{A}$, leads to general left-handed distributivity. The condition for general right-handed distributivity is $\overline{\mathfrak{M}}\mathfrak{A}\overline{\mathfrak{M}}^{-1} = \mathfrak{A}$, where $\overline{\mathfrak{M}}$ is the adjoint group of $\mathfrak{M}$. Von Staudt's algebra of points on a line is obtained by letting $\mathfrak{A}$ be the group of all parabolic projectivities having a common double point M, and letting $\mathfrak{M}$ be the group of all projectivities having two common double points M and O, where O is the

identical element with respect to $\mathfrak{A}$. The author discusses several other examples which lead to new analytic representations of certain groups of projective and birational transformations.

22. In his second paper, Professor Young employs the methods developed in his first paper to define a point algebra in a space S_n of n dimensions as follows: Multiplication is defined by the group $\mathfrak{M}$ of collineations in S_n leaving the vertices $O, U_1, U_2, \cdots, U_n$ of a complete $(n+1)$-point invariant, $a \cdot b$ being by definition the point into which b is transformed by the transformation M_a of $\mathfrak{M}$ which transforms the unit point 1 into a. Addition is defined by the group $\mathfrak{A}$ of elations in S_n leaving fixed every point of the S_{n-1} determined by $U_1, \cdots, U_n$, $a+b$ being the point into which b is transformed by the transformation A_a of $\mathfrak{A}$ which transforms the zero point $O = 0$ into a. Multiplication and addition are then associative and commutative, and multiplication is distributive with respect to addition. In fact, the points of S_n form a field with respect to these two operations, the exceptional elements being the points of the $n(n-1)$-spaces of the fundamental $(n+1)$-point $O, U_1, \cdots, U_n$ for multiplication and the points of the S_{n-1} determined by $U_1, \cdots, U_n$ for addition. The author then considers the group $\mathfrak{A}_n$, of $3n$ parameters, consisting of all transformations $x' = \dfrac{ax+b}{cx+d}$, $ad-bc$ not a divisor of zero. $\mathfrak{A}_1$ is then the general projective group on a line; $\mathfrak{A}_2$ is the group of all quadratic transformations in a plane having two fundamental points U_1 and U_2 in common, and is therefore equivalent to the group of direct circular transformations in a plane first discussed by Möbius; $\mathfrak{G}_3$ is a group of Cremona transformations in S_2 of order 3 having three fundamental points U_1, U_2, U_3 in common; in general $\mathfrak{G}_n$ is a group of Cremona transformations of order n in S_n. The representation of the group given above makes it possible to apply methods of the projective geometry on a line to the geometry defined by $\mathfrak{G}_n$. The elementary curves of the geometry, the so-called C-curves, are the curves equivalent under $\mathfrak{G}_n$ to the straight line through 0 and 1. There is one and only one such curve through any 3 points a, b, c, provided no two of the latter are coplanar with two of the points U_i. The class of C-curves contains all the normal

curves of degree n in S_n, passing through the n points U_i $(i = 1, \cdots, n)$ and certain normal curves of lower orders. For example, for $n = 3$ the C-curves consist of all twisted cubics through U_1, U_2, U_3, of all conics through one of the points U_i and meeting the opposite side of the triangle $U_1U_2U_3$, and of all straight lines in S_3 not on the plane $U_1U_2U_3$. Any transformations of $\mathfrak{G}_n$ of general type has 2^n double points, which are associated in a definite way into 2^{n-1} pairs. Among the theorems derived may be mentioned the following: Any involution (of general type) in $\mathfrak{G}_n$ leaves invariant every C-curve through every pair of associated double points, and only these. If a transformation of $\mathfrak{G}_n$ leaves one C-curve invariant, it leaves one, and if the transformation is non-involutoric only one, such curve through every point not a double point invariant; etc. The author also discusses the extension of the group $\mathfrak{G}_n$ by the adjunction of transformations of another kind, which reduces in the case $n = 2$ to the introduction of the indirect circular transformations.

23. A series is absolutely convergent, if the series formed from the absolute values of its terms converges; unconditionally convergent, if it converges for every order in which its terms are arranged. The two definitions are equivalent for convergent series. In this note, Dr. Silverman proposes to inquire whether similar notions are possible in the theory of summable series.

The notion of absolute convergence is obviously not generalizable to the case of summable series,* if we restrict† ourselves to those definitions for which it is true that whenever $\lim(S_n) = s$ or ∞, then $G(S_n) = s$ or ∞ respectively, where lim stands for the ordinary limit, and G for the generalized limit. That the notion of unconditional convergence, too, cannot be generalized to the case of summable series, if we restrict ourselves to the same definitions as above, results from the following theorem: If corresponding to every arrangement of the terms of a series there is some definition of summability which gives it a value (possibly different in each case), then the series is absolutely convergent.

* Borel has used the term absolute summability, but in a different sense from that considered here. See Leçons sur les Séries divergentes, p. 99.

† This restriction excludes such definitions as that of absolute summability given by Borel; but this definition of Borel's is not a true generalization of convergence, as has been pointed out by Hardy, *Quarterly Journal*, vol. 35 (1903), p. 25.

24. In this note Dr. Silverman considers some simple sufficient conditions for Cesàro summability, similar to some of the well-known tests for convergence.

25. Professor Fite calls attention to a collineation group in n variables that seems not to have been noticed before. It is of order $n^3b^{a-1}d^{c-1}/f$, where $n = ab = cd$, a and b relatively prime, c relatively prime with both a and d, $a < n$, $c < n$, and f the greatest common divisor of $c - a$ and d. It contains an invariant abelian subgroup of index n.

26. Professor Eisenhart gives the following analytical definition of a congruence of minimal lines in euclidean four-space: Given a congruence C of non-minimal lines in ordinary space, each line being defined by the coordinates (x_0, y_0, z_0) of the point P_0 in which the line meets a surface of reference S referred to curvilinear coordinates u, v and by its direction cosines X, Y, Z, where x_0, y_0, z_0 and X, Y, Z are one-valued continuous functions of u and v, admitting continuous first derivatives; if x, y, z are the coordinates of a point P on this line and t the distance P_0P, then the equations

$$x = x_0 + tX, \quad y = y_0 + tY, \quad z = z_0 + tZ, \quad w = it$$

define a congruence Γ of minimal lines in four-space of the most general type. The condition that a one-parameter system of these lines, determined by a relation $f(u, v) = 0$, be tangent to a curve, i. e., developable two-spread, is that the corresponding ruled surface of the congruence C, which is defined by the first three of the above equations, be a developable surface, and that it meets S in an orthogonal trajectory of the generators of this developable. When C is a normal congruence, and only in this case, through each line of Γ there pass two developables of the congruence. (This is the particular type of minimal congruences recently considered by Eiesland in the *Transactions*.) However, any surface S and a family of curves upon it leads to a congruence Γ such that through each line of Γ there passes one developable. But for the general congruence Γ none of the ruled two-spreads is developable. The paper deals with other properties of congruence Γ.

F. N. COLE,
Secretary.

THE FIFTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE fifth regular meeting of the Southwestern Section of the Society was held at Washington University, St. Louis, Missouri, on Saturday, December 2, 1911. About twenty-five persons were present, including the following members of the Society:

Professor W. C. Brenke, Dr. A. R. Crathorne, Professor Ellery W. Davis, Dr. Otto Dunkel, Professor H. C. Harvey, Professor E. R. Hedrick, Professor G. O. James, Professor O. D. Kellogg, Professor A. M. Kenyon, Mr. W. C. Krathwohl, Dr. S. Lefschetz, Professor H. L. Rietz, Professor W. H. Roever, Professor J. N. Van der Vries, Professor C. A. Waldo.

The morning session opened at 10 A.M. and the afternoon session at 3 P.M., Professor Roever presiding. Lawrence, Kansas, was decided upon as the next place of meeting and the following program committee was elected: Professors Van der Vries (chairman), Davis, and Kellogg (secretary). The members presentt atended a smoker at the University Club on the evening before the meeting and dined together between the sessions in the dining room of Tower Dormitory.

The following papers were presented at this meeting:

(1) Professor R. D. CARMICHAEL: "Generalizations of Euler's φ-function with applications to abelian groups."

(2) Dr. OTTO DUNKEL: "A necessary condition for the reality of the roots of an algebraic equation."

(3) Professor O. D. KELLOGG: "Note on periodic functions with derivatives of all orders."

(4) Dr. S. LEFSCHETZ: "Two theorems on conics."

(5) Professor M. B. WHITE: "The dependence of the focal point on curvature in space problems in the calculus of variations."

(6) Professor IVA ERNSBERGER: "The imaginary parabola" (preliminary communication).

(7) Professor HEDRICK and Dr. INGOLD: "Axioms of line geometry."

(8) Professor W. H. ROEVER: "The deviation of falling bodies for a distribution not of revolution."

(9) Professor E. R. HEDRICK: "The concept of limit."

(10) Professor G. O. JAMES: "Comparison of methods of adjusting outstanding differences in the motions of the four inner planets."

(11) Dr. A. B. FRIZELL: "On a subset of the terms in the infinite determinant having the potency of the continuum."

Professor Ernsberger was introduced by Professor Davis. In the absence of the authors Professor Carmichael's paper was read by title and the papers of Professor White and Dr. Frizell were presented by Professor Van der Vries. Abstracts of the papers follow.

1. Among the many generalizations which have been given of Euler's φ-function, particular mention may be made of those due to Schemmel, Lucas, Vahlen, Cohen, and Zsigmondy. The last (*Monatshefte für Mathematik und Physik*, volume 7 (1896), pages 185–289) especially, is one of far-reaching importance. It comes out as an application of a general theory of abelian groups, which Zsigmondy develops. The object of Professor Carmichael's paper is to define in a new way Zsigmondy's generalization of the φ-function, and to develop its fundamental properties from the point of view of the new definition. As immediate corollaries of this theory of the generalized φ-function, one obtains Zsigmondy's fundamental results in the theory of abelian groups by means which are essentially simpler than those employed by Zsigmondy.

2. A necessary condition that all the roots of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ be real is that the series of determinants of even order

$$\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix}, \quad \cdots$$

have signs alternately $-$ and $+$. This result which is obtained by Dr. Dunkel by the methods employed in a previous paper in the *Annals of Mathematics*, series 2, volume 10 (1908), page 46, bears some analogy to a sufficient condition already given by Professor E. B. Van Vleck in the same journal, series 2, volume 4 (1903), page 191.

3. If $f(x)$ together with its derivatives of all orders has the period 2π, then either $\max|f^{(k)}(x)| > Ae^{\lambda k}$ for all but a finite number of values of k, A and λ being positive constants, or else $f(x)$ is a terminating Fourier series. Professor Kellogg's paper gives a simple proof of this fact, and mentions some inferences.

4. In this paper Dr. Lefschetz treats of two correlated theorems on conics. (1) When two complete quadrilaterals, circumscribed to the same conic S have a common diagonal, their vertices exterior to it are on a conic Γ. Let D be the common diagonal, A, B, A', B' the vertices on it. It is shown both analytically and synthetically that for a given conic the vertices A, B are conjugate elements of an involution on D, and conversely that to each involution on D there corresponds one and only one conic Γ. These theorems can be deduced easily from others given by Poncelet, although the proofs of the present paper are completely different. It is further shown that the system S, Γ is projectively ∞^1, and the relation satisfied by the invariants of the two conics is also given. (2) When the tangents at three of the six common points of a conic and a triangle go through the opposite vertices, they all do, the triangle being then self-polar with respect to the conic. Here also both analytic and synthetic proofs are given, and a third proof based upon projective considerations. This theorem is equivalent to the following well known proposition in the geometry of rational plane quartics: if a rational plane quartic has three flecnodes, it has necessarily three biflecnodes. The paper will be offered for publication to the *Annals of Mathematics*.

5. If an arc C_{01} which joins a space curve L and a fixed point 1 minimizes the integral $J = \int f(x, y, y', z, z')dx$ with respect to other curves joining L with 1, there will in general be a focal point 2 on the curve of which C_{01} is a part at which this minimizing property ceases. Professor White shows that the position of the focal point is a function of the curvature of L at 0 and discusses its properties by means of the second variation. The paper will be offered to the *Transactions*.

7. In this paper Professor Hedrick and Dr. Ingold give a system of axioms for line geometry based upon the fundamental ideas of line and intersection. They are able with a com-

paratively simple set of assumptions, to prove the fundamental propositions of line geometry. The assumptions may also be made the basis of a treatment of projective geometry by defining the words point, plane, etc., in terms of the fundamental ideas mentioned above.

8. In order to be able to compare the results of experiment with those of theory, the following definitions are taken: Let P_0 denote the point from which the body falls, c the path of the falling body, and d the locus of plumb-bobs of all plumb-lines which are supported at P_0. The curve d pierces a level surface in a point D. Let π denote the tangent plane to this level surface at D, and C the point in which the curve c pierces the plane π. Let α and β denote the east-and-west and the north-and-south lines respectively of the plane π which pass through D. The distance of C south of α is the southerly deviation of the falling body and that of C east of β is the easterly deviation of the falling body. If h represents the height of P_0 above π and W the potential function of the field of force in which the plumb-line is in equilibrium, it is possible to express the deviations just defined in terms of h and the first and second derivatives of W at P_0. These expressions are deduced in Professor Roever's paper. The quantities h and g (the only first derivative of W which enters) can be measured. The second derivatives of W which enter can be found experimentally by a method due to Baron Eötvös. (Encyklopädie der mathematischen Wissenschaften, Band VI, 1, B, Heft 2, § 23, page 166.) Hence it is not necessary to assume a mathematical expression for W. Since experiments have been, and are likely to be, made in localities where the local deviation of the gravitational field is considerable (due to the proximity of mountains) the advantage of this method is apparent.*

9. In the paper presented by Professor Hedrick it is shown that a material simplification of the system of axioms proposed by Steinhaus in a recent number of the *Mathematische Annalen* for the concept "limit" leaves that concept defined uniquely on convergent sequences, and permits extension to divergent sequences in the Cesàro sense. It is shown that the concept

* The southerly deviation for a distribution of revolution has already been treated by Professor Roever in a paper in the *Transactions*, vol. 12, No. 3, pp. 335–353.

"limit" may be defined for general systems of objects in an analogous manner, without excluding the possibility of special examples of the Cesàro type.

10. The nodes and perihelia of the four inner planets, notably Venus and Mars, present certain unexplained motions in the Newtonian mechanics. The note of Professor James compares the secular changes in the elements of these planets produced by the uniform rotation of the empirical about the inertial system of reference with the corresponding changes brought about by the use of the Minkowskian law of attraction instead of the Newtonian.

11. In this paper Dr. Frizell shows that a one-to-one relation exists between the continuum and a set of terms in the expansion of an infinite determinant whose elements are restricted to the principal diagonal and two adjacent diagonals.

O. D. KELLOGG,
Secretary of the Section.

SERIES OF LAPLACE'S FUNCTIONS.

BY PROFESSOR B. H. CAMP.

(Read before the American Mathematical Society, October 28, 1911.)

THE most important theorem on the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions there stated are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through this point.

The object of the present paper is to correct an error in Jordan's theorem, and to furnish new conditions sufficient for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be "uniform with respect to all great circles through the point." His error is discussed in a remark following Corollary 2.

In Theorem 1, I have corrected Jordan's theorem and extended it to the case where any or all of the conditions may fail on a null set of great circles through the point. Theorem 2 is a new theorem, and replaces the conditions of continuity of the function and uniformity of the variations, in Theorem 1, by the different requirement that there exist a small circle about the point considered such that within it the given function is, along every great circle through the point, an indefinite integral of another function which has an absolutely convergent double Lebesgue integral in its domain of definition. A null set of great circles may be neglected in this theorem also.

All the integrals used are L-integrals, i. e., integrals in the sense of Lebesgue. Limited variation may be understood to refer either to the definition of Jordan or to that of Pierpont; the theorems are true for both definitions.

Suppose the function $f(p)$ to be defined and limited and to have a double Lebesgue integral on the surface of a unit sphere, p being a point of the surface, and suppose it to be required to develop this function in a series of Laplace's functions which shall be valid at a fixed point p_0. We learn from Jordan's work, subject to certain transformations which are permissible* in the present case, that the validity of the formal development depends on the convergence to $\Phi(x_0)$ of

$$(1) \qquad S_n = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x, \mu)\,[X'_{n+1}(x) + X'_n(x)]dx,$$

where $\Phi(x, \mu)$ is the form assumed by $f(p)$ when the coordinates (x, μ) of p are chosen as stated below, and X'_1 is the derivative with respect to x of X_n, the function of Legendre. Here p_0 is taken as the north pole of the sphere, x as the cosine of the colatitude of p, and μ as the longitude of p with respect to some fixed meridian $\mu = 0$. Evidently $f(p_0) = \Phi(x_0) = \Phi(1)$, and is independent of μ. Now since

$$\int_{-1}^{+1} (X'_{n+1} + X'_n)\,dx = 2,$$

it follows that

$$\Phi(x_0) = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x_0)\,(X'_{n+1} + X'_n)dx.$$

* Lebesgue, *Annales l'Ecole Norm. Sup.*, ser. 3, vol. 27 (1910), pp. 447–50.

Subtracting this from (1), we have, placing $\psi(x, \mu) = \Phi(x, \mu) - \Phi(x_0)$,

$$(2) \quad 4\pi(S_n - \Phi(x_0)) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(x, \mu)\,(X'_{n+1} + X'_n)dx,$$

and the validity of the expansion to be studied depends on the convergence of (2) to zero with $1/n$.

LEMMA 1. *If $f(x)$ has limited variation of value V in the interval (a, b), and $\phi(x)$ is absolutely L-integrable in this interval,*

$$\left| \int_a^b f\phi \right| \leq (V + \max |f|) \max_{\lambda} \left| \int_a^{\lambda} \phi \right|, \ \lambda \text{ in } (a, b).$$

This is readily deduced from a theorem of Lebesgue's (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), page 37).

THEOREM 1. (1°) *Let $\Phi(x, \mu)$ be defined and limited and have a double* L-integral in the rectangle $R = (-1 \leq x \leq 1, 0 \leq \mu \leq 2\pi)$;* (2°) *let it be, at $x = 1$, a function of x uniformly continuous with respect to μ;* (3°) *let it have limited variation in $(-1, 1)$, and let the value of this variation be limited with respect to μ.* (4°) *Let the variation of Φ in $(1 - x, 1)$ be, at $x = 1$, a function of x uniformly continuous with respect to μ. Finally, any or all of these conditions may fail for a null of set μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. We will establish the theorem by showing that (2) converges to zero with $1/n$. Let ϵ be > 0 and arbitrary, and select h so that by 2°

$$(3) \qquad \max |\psi| < \epsilon \text{ in } (1 - h, 1)$$

for all μ's uniformly, except perhaps a null set; and so that by 4°, for the function ψ,

$$(4) \qquad V(1 - h, 1) < \epsilon$$

for all μ's uniformly, except perhaps a null set. We may

*It is known that

$$\int_R \psi(X'_{n+1} + X'_n) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(X'_{n+1} + X'_n)dx.$$

A slightly more general condition would suffice, but the difference is trivial.

write (2) in the form

$$\int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n)dx$$

(5)

$$+ \int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(X'_{n+1}+X'_n)dx = \int_0^{2\pi} (I)\, d\mu + \int_0^{} \quad (II)d\mu.$$

By 3° and Lemma 1,

$$|\,\mathrm{II}\,| \leq (V(1-h, 1) + \max|\,\psi\,|\,) \max_{\lambda} \left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right|,$$

λ in $(1-h, 1)$, for all μ's, except perhaps a null set. But

$$\left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right| = |\,2 - X_{n+1}(\lambda) - X_n(\lambda)\,| < 4,$$

and therefore

$$|\,\mathrm{II}\,| < 4(\epsilon + \epsilon), \tag{6}$$

by (3) and (4), for all μ's uniformly, except perhaps a null set. Similarly

$$|\,I\,| \leq (V(-1, 1-h) + \max|\,\psi\,|\,) \max_{\lambda} \int_{-1}^{\lambda} (X'_{n+1} + X'_n)dx,$$

λ in $(-1, 1-h)$, for all μ's except perhaps a null set. The integral here equals $X_{n+1}(\lambda) + X_n(\lambda) - 0$, and Jordan has shown (page 236) that this expression converges to zero with $1/n$ uniformly with respect to λ in $(-1,\ 1-h)$, provided only that $0 < h \leq 2$. Since, by 1° and 3°, there exists an M such that $V + \max|\,\psi\,| < M$, we have now shown that there exists an n_ϵ, independent of μ, so that

$$|\,I\,| < M\epsilon, \quad n > n_\epsilon, \tag{7}$$

for all μ's uniformly except perhaps a null set. This with (6) and (5) shows that

$$\left| \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi\,(X'_{n+1} + X_n')dx \right| < 2\pi(8\epsilon + M\epsilon), \quad n > n_\epsilon;$$

which proves the theorem.

REMARK. *In order that* 4° *may be fulfilled, it is sufficient*

that Φ be, with respect to μ, uniformly continuous at $x = 1$, and that in $(-1, 1)$ it have limited variation which is "uniform with respect to μ" (i. e., that $\lim_{d=0} V_D = V$ uniformly with respect to μ, where V is the value of the variation in $(-1, 1)$, and V_D the usual sum function for the arbitrary division D of norm d).

COROLLARY 1. *The same conclusion follows if, in place of 3° and 4°, we have the condition that Φ be a monotone increasing function of x.*

For 3° is then satisfied because $V \leq \max |\psi|$ in the rectangle R. Moreover, if $d \leq 2$, $V - V_D = 0$, for all μ's, except perhaps a null set. Therefore, by the Remark above, 4° is also satisfied.

COROLLARY 2. *If $f(p) = f_1(p) - f_2(p)$, where f_1 and f_2 both satisfy the conditions of Corollary* 1, *the same conclusion follows.*

It is this last corollary that Jordan has actually proved, except that he may not neglect a null set of meridians, and that he should have stated that f must be limited on the sphere as well as along each meridian. (Cf. page 230, "D'autre part $\psi(b - 0)$ est fini. . . ." At the bottom of page 251 he desires to have the convergence uniform with respect to μ, and therefore ψ "fini" uniformly with respect to μ.) Having proved this corollary, however, it is not permissible to conclude, as Jordan does, that the theorem will be true without the restrictions stated in 4° and in the last part of 3°. As he remarks in § 220, his theorems are proved for monotone functions, and stated for functions of limited variation. This is a proceeding which is often allowable, because of the well known fact that any function of limited variation is the difference of two monotone increasing functions, but it may lead to error unless the two monotone functions may be made to obey the restrictions imposed on the given function. As a matter of fact, in the case before us, it is not difficult to construct a limited, integrable function, which satisfies the conditions imposed by Jordan, but is not expressible as the difference of two monotone increasing functions each of which separately obeys the same conditions. In a remark appended to a fundamental convergence theorem (§ 222, page 230) Jordan recognizes this difficulty, but he ignores his remark when he afterwards applies his theorem to Laplace's functions. He has actually found, then, the conditions which

certain components of the given function should satisfy, not those which are to be satisfied by the function itself.

LEMMA 2. *If of two functions, of one or of two variables, one is absolutely L-integrable in a limited field, and the other is limited, their product is absolutely L-integrable in the same field.**

LEMMA 3. *Let A be any limited, measurable field in the x, y plane. In order that the integral*

$$\int_A f(x, y)\, \phi(x, y;\ n)$$

may converge to zero with $1/n$ for all absolutely L-integrable functions f defined in A, it is necessary and sufficient that there exist an M and an n_M so that $|\phi| < M$, $n > n_M$, except perhaps at a null set of points, and that $\int_r \phi \doteq 0$ with $1/n$ for each rectangle r of sides parallel to the axes x, y.

This may be proved in a manner analogous to the proof given by Lebesgue for the corresponding theorem in one dimension (*Annales de la Faculté de Toulouse*, loc. cit., page 52)† by the use of another theorem proved by him in *Annales de l'Ecole Normale Supérieure*, series 3, volume 27 (1910), page 374.

THEOREM 2. *Using the notation of Theorem* 1, (1°) *let $\Phi(x, \mu)$ be defined, limited, and have a double L-integral in R.* (2°) *In $(1 - h, 1)$, for some $h > 0$ and independent of μ, let Φ be the indefinite integral, with respect to x, of another function $\theta(x, \mu)$, which has an absolutely convergent, double L-integral in its field of definition.* (3°) *Let Φ have limited variation in $(-1, 1)$ the value of which is limited with respect to μ. Finally, any or all of these conditions may fail for a null set of μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. As in Theorem 1, we may write (2) in the form (5), and, by virtue of 1° and 3° of our hypothesis, the first of these integrals,

$$(8) \qquad \left| \int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n) dx \right| < 2\pi M \epsilon, \quad n > n_\epsilon.$$

* Lebesgue, loc. cit., p. 374.

† Cf. also Hobson, *Proc. London Math. Soc.*, ser. 2, vol. 6 (1908), p. 355.

As for the second, by 2° and a theorem due to Lebesgue,*

$$(9)\qquad \int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx=[\psi(X_{n+1}+X_n)]_{1-h}^{1}$$

$$-\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx=-\psi(1-h,\,\mu)\cdot(X_{n+1}(1-h)+X_n(1-h))$$

$$-\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx,\quad \text{since } \psi(1,\,\mu)=0.$$

We may now write the second integral of (5) in the form

$$(10)\quad \int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx=-\int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(1-h,\,\mu)$$

$$\cdot(X_{n+1}(1-h)+X_n(1-h))dx-\int_0^{2\pi}d\mu\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx,$$

provided two of these integrals exist. To show that they do exist let us write R_h for the rectangle $1-h\leq x\leq 1$, $0<\mu\leq 2\pi$. Then, by 2° and Lemma 2, $\int_{R_h}(X_{n+1}+X_n)\theta$ exists, and by a recent theorem of Hobson†

$$(11)\qquad \int_{R_h}(X_{n+1}+X_n)\theta=\int_0^{2\pi}d\mu\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx.$$

We knew before that the first integral of (10) exists.

We now employ again that portion of Jordan's work used to derive (7) in Theorem 1, and learn that

$$|\,X_{n+1}(1-h)+X_n(1-h)\,|<\epsilon,\quad \text{if } n>n_\epsilon,$$

and by 1° we know that $|\,\psi(1-h,\,\mu)\,|<M$; and these inequalities hold for all μ's uniformly, except perhaps a null set. By virtue of these and (11), (10) shows us that

$$(12)\qquad \left|\int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx\right|<2\pi\epsilon M+\left|\int_{R_h}(X_{n+1}+X_n)\theta\right|,$$

if $n>n_\epsilon$. Comparing this with (8) we find that in order to establish the theorem it only remains to show that

* Loc. cit., p. 46.

† *Proc. Lond. Math. Soc.*, ser. 2, vol. 8 (1910), p. 30.

$$\int_{R_h} (X_{n+1} + X_n)\theta \doteq 0 \quad \text{with} \quad 1/n. \tag{13}$$

To do this we use Lemma 3, θ being an absolutely L-integrable function by hypothesis, and $| X_{n+1} + X_n |$ being ≤ 2 for all values of n.

Let r be an arbitrary rectangle in R_h whose sides are parallel to the μ, x axes. It only remains to show that

$$\int_r X_{n+1} + X_n) \doteq 0 \quad \text{with} \quad 1/n. \tag{14}$$

By two well known formulas,* if $| k | < 1$,

$$\int_k^1 X_n(x)dx = \frac{(1-k^2)(X_n'(k))}{n(n+1)},$$

and

$$X_n'(k) = \frac{nkX_n(k) - nX_{n-1}(k)}{k^2 - 1}.$$

Therefore

$$\left| \int_k^1 X_n(x)dx \right| < \frac{k+1}{n+1}, \quad \text{since} \quad | X_n(k) | < 1.$$

This integral does not depend on μ, and k depends only on r. (14) follows, and the theorem is proved.

The necessary and sufficient conditions that a function be an indefinite integral are given by Lebesgue in the *Rendiconti dell' Accademia dei Lincei*, volume 16 (1907), 1st semester.

The following function satisfies all the conditions of Theorem 2, but not 2° of Theorem 1:

$$\Phi = \frac{1}{\sqrt{\mu}}(x-1) \quad \text{in} \quad 1 - \frac{\sqrt{\mu}}{3} \leq x \leq 1, \text{ if } \mu \neq 0$$

$$= -\frac{1}{3} \text{ elsewhere, if } \mu \neq 0$$

$$= 0, \text{ if } \mu = 0.$$

WESLEYAN UNIVERSITY,
October, 1911.

* Cf., e. g., Byerly, Fourier's series and spherical harmonics, pp. 172, 180.

ON A NEW MIXED PROBLEM OF THE PARTIAL DIFFERENTIAL EQUATION OF TELEGRAPHY.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society, September 13, 1911.)

DURING the last twenty years the question of the propagation of electric waves in wire lines or cables has been the subject of a very great number of researches, on the one hand on account of its practical importance in connection with telephony, rapid telegraphy, wireless telegraphy, and the transmission of energy by means of alternating currents, and on the other on account of its purely mathematical interest and of the great increase in our knowledge of partial differential equations of the hyperbolic type. In all these researches the point of view has been very different in the different cases and may be roughly classified as follows. The electrical engineer, whether interested in transmission, telephony, or wireless, has been mainly concerned with phenomena depending upon simple harmonic functions of the time, that is, with standing wave phenomena after the steady state of oscillation has been established. For these the methods of trigonometric series suffice perfectly, and with them we are not here concerned. In these cases the line is of finite length, and we deal with series of functions. On the other hand the mathematician has been interested in problems connected with propagation in a line of infinite length, especially as depending on the data of the initial state of the line. For this case instead of infinite series we should use definite integrals of Fourier, but the method of Riemann has furnished a more powerful and interesting method of attack, which should interest the physicist on account of the clearness with which the results may be interpreted. It would be impossible to cite all the important memoirs that have appeared on this subject, but it would be equally impossible to omit the names of Heaviside, Picard, Poincaré, Boussinesq, Goursat, Brillouin, and Hadamard.

The first appearance of the so-called telegraphist's equation is in a paper by Kirchhoff,* which is remarkable in that it

* Kirchhoff, "Ueber die Bewegung der Elektricität in Drähten," *Pogg. Ann.*, vol. 100 (1857); Ges. Abh., p. 131.

appeared before Maxwell's theory of electricity was invented. Twenty years later the equation was first given on the basis of that theory by Heaviside, in a very remarkable paper,* which, like Kirchhoff's, remained unnoticed. Both Kirchhoff and Heaviside used the method of Fourier's series of what are now called normal functions. The treatment of the infinite line began with Poincaré,† who employed the Fourier integral, Boussinesq,‡ who gave a new modification of the method used by Poisson for the wave equation, and Picard,§ who used the method of Riemann.

All these writers were interested in the propagation of the waves as determined by the initial data in the whole line, that is, in the so-called Cauchy problem. Later appeared papers by Brillouin,‖ in which propagation in three dimensions, and the effect of sources of waves was discussed. But in none of these papers was the problem touched which is often of great importance to the experimental physicist, namely the effect upon the waves of the terminal apparatus, composed of any arrangement of coils, condensers, and resistances, which are connected to one or both ends of a line of finite length, and to which is applied at some point an electromotive force which varies according to a given function of the time. It is true that the influence of the terminal apparatus was completely discussed by Heaviside¶ in a very powerful paper written in 1882, but published first in 1892, and in a series of others, well characterized by M. Brillouin, as "une prodigieuse série de mémoires à la fois condensés et touffus," in which the method is throughout that of development in series of normal functions. It is for this reason that Heaviside remarks: "it is an enormous and endless subject, admitting of infinite development." It is believed that for this reason anything that will throw light upon or simplify the matter will be of interest.

* Heaviside, "On the extra current," *Phil. Mag.*, Aug., 1876; Electrical Papers, vol. I, p. 53.

† Poincaré, "Sur la propagation de l'électricité," *Comptes Rendus*, vol. 117 (1893), p. 1027.

‡ Boussinesq, "Intégration de l'équation du son pour un fluide indéfini," *Comptes Rendus*, vol. 118 (1894), p. 162.

§ Picard, "Sur l'équation aux dérivées partielles qui se rencontre dans la théorie de la propagation de l'électricité," *Comptes Rendus*, vol. 118 (1894), p. 16.

‖ Brillouin, "Propagation dans les milieux conducteurs," *Comptes Rendus*, vol. 136 (1903), p. 667, 16 mars; p. 746, 23 mars.

¶ Heaviside, "Contributions to the theory of the propagation of current in wires," Electrical Papers, vol. 1, p. 141. "On the self-induction of wires, *Phil. Mag.*, 1886–7; Papers, vol. 2, p. 168.

If I denote the current, V the potential at a point at a distance z from the origin at the time t, the equations of propagation are

$$(1) \qquad L\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial z}, \qquad K\frac{\partial V}{\partial t} + SV = -\frac{\partial I}{\partial z},$$

where L is the self-inductance, K the capacity, R the resistance, and S the leakage conductance of the line, all referred to unit of length. If we change the units of length and potential, and remove an exponential factor, putting

$$(2) \qquad \begin{aligned} I &= e^{-qt}u, \qquad V = -\sqrt{\frac{L}{K}}e^{-qt}v, \quad 2q = \frac{R}{L} + \frac{S}{K}, \\ x &= \sqrt{KL}\cdot z, \quad y = t, \qquad\qquad 2p = \frac{R}{L} - \frac{S}{K}, \end{aligned}$$

we reduce to the simple form

$$(3) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} - pv, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + pu,$$

from which we obtain at once

$$(4) \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + p^2 u = 0,$$

which equation is also satisfied by v. This is the telegraphist's equation which by the substitution $px = \xi + \eta$, $py = \xi - \eta$ is reduced to the normal form

$$(5) \qquad \frac{\partial^2 u}{\partial \xi \partial y} + u = 0.$$

When $p = 0$ we have the simple equation for the motion of a stretched string

$$(6) \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

This, called by Heaviside the distortionless case, was that treated by Kirchhoff. The researches of Picard, Goursat, and Hadamard have been on the more general equation

$$(7) \qquad \frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0,$$

to which most of the results here reached will apply.

The ordinary, or Cauchy problem, consists in determining the solution by giving the values of u and $\partial u/\partial y$ for the whole line $y = 0$. It has been shown by Darboux, Goursat, and Picard, that instead we may give the values of u alone, not giving the normal derivative, along *two* lines meeting at an angle.* It has been remarked by Hadamard† this is not the problem that interests the physicist, but that, as in the present case, we have Cauchy's data for only a portion of the line $y = 0$, while at the end $x = 0$ we have a different condition, not being able to give both Cauchy data, but only one, namely the values of u. This he terms the *mixed* problem.

We are here concerned with a line infinite in one direction‡ which ends at the point $x = 0$, but instead of giving either u or its normal derivative along the line $x = 0$, $y \geqq 0$, we give a relation between them in the form of a linear differential equation in y. It may be easily shown that if we have any combination of coils, with self or mutual inductance and resistance, with condensers, connected to the end of the line, with an electromotive force applied somewhere in the combination, the effect will be to make

$$a_0 u + a_1 \frac{\partial u}{\partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + \cdots + a_m \frac{\partial^m u}{\partial y^m} + b_0 v + b_1 \frac{\partial v}{\partial y} + \cdots + b_n \frac{\partial^n v}{\partial y^n} = \psi(y), \quad (x = 0,\ y \geqq 0), \tag{8}$$

from which, eliminating v by the first of equations (3) and an equation (18) or (33), deduced below, we get a differential relation between u and the normal derivative $\partial u/\partial x$. Speaking mechanically, the usual end conditions have been $u = 0$ for a string fixed at the end, or $\partial u/\partial x = 0$ or $\partial u/\partial x - hu = 0$ which have been used for other problems. If the string is fastened, however, to a movable point on an apparatus possessed of elasticity, inertia, and dissipation, the condition will be as stated in (8). The writer was led to consider this condition in 1893, from electrical experiments that he was then making, but on account of lack of mathematical knowledge, the question was laid aside. It appears that the method used by Hadamard leads without difficulty to the result.

* See also Mason, New Haven Mathematical Colloquium.

† Hadamard, "Sur un probleme mixte aux dérivées partielles," *Bull. de la Soc. Math. de France*, vol. 31 (1903), p. 208.

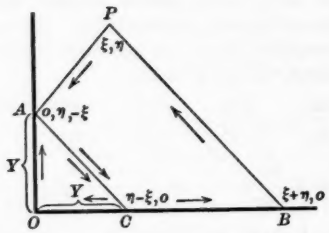
‡ If the second end is at a finite distance, the reasoning here given may be applied to it.

It seems intuitively evident to the physicist that in the present case neither u nor the normal derivative can be given for $x = 0$, but that both are completely determined by the condition (8). The calculation shows this to be the case. For simplicity let us first consider the case of no distortion, $p = 0$. We have then, if we give the initial distribution of current and potential

$$(9) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

$$(10) \quad u = f(x), \quad v = g(x), \quad \frac{\partial u}{\partial y} = g'(x), \quad y = 0, \quad x \geqq 0.$$

Let P be any point with coordinates ξ, η from which two characteristics meet the axes in A, B, and from A draw the



characteristic meeting the x-axis in C. The fundamental equation of Riemann's method is

$$(11) \qquad \int G\left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\} - u\left\{\frac{\partial G}{\partial x}dy + \frac{\partial G}{\partial y}dx\right\} = 0$$

for a closed contour, where u and G are solutions of the given equation and its adjoint respectively (here the same equation). In the present simple case the function G is constant and equal to unity, so that we have

$$(12) \qquad \int\left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\} = 0.$$

Applying (12) to the closed quadrilateral $PACBP$, we have

$$PA, \quad dx = dy, \quad \int_P^A du = u_A - u_P,$$

$$(13) \quad \left.\begin{matrix} BP \\ AC \end{matrix}\right\}, \quad dx = -dy, \quad -\int_B^P du = u_B - u_P, \quad -\int_A^C du = u_A - u_C,$$

$$CB, \quad dy = 0, \quad \int_C^B \frac{\partial u}{\partial y} dx = \int_C^B \frac{\partial v}{\partial x} dx = v_B - v_C.$$

Adding these results

$$(14) \qquad 2u_P = 2u_A + u_B - u_C + v_B - v_C.$$

Accordingly it is necessary to determine only the value of u_A, when we have the result as in Hadamard's case.

If we apply the formula (12) to the triangle AOC, we have

$$(15) \qquad 0 = \int_o^A \frac{\partial u}{\partial x} dy + u_A - u_C + v_O - v_C,$$

so that

$$(16) \qquad u_A = u_C + v_C - v_O - \int_o^Y \left(\frac{\partial u}{\partial x}\right)_{x=0} dy.$$

Inserting in (14), we have

$$(17) \quad u_P = \frac{1}{2}(u_B + u_C) + \frac{1}{2}(v_B + v_C) - v_O - \int_o^Y \left(\frac{\partial u}{\partial x}\right)_{x=0} dy,$$

so that the problem is merely shifted to determining the normal derivative $\partial u/\partial x$ along $x = 0$. Differentiating (16) by Y,

$$(18) \qquad \frac{\partial u(Y)}{\partial Y} = f'(Y) + g'(Y) - \left(\frac{\partial u}{\partial x}\right)_{x=0}.$$

Differentiating the equation (8) and substituting for $\partial v/\partial y$ its value

$$(19) \qquad \frac{\partial u}{\partial x} = f'(y) + g'(y) - \frac{\partial u}{\partial y},$$

we have the linear differential equation

$$(20)\quad (a_0 - b_0)\frac{\partial u}{\partial y} + (a_1 - b_1)\frac{\partial^2 u}{\partial y^2} + \cdots = \psi'(y) - b_0(f'(y)+g'(y)) - b_1(f''(y)+g''(y)) + \cdots.$$

Solving this with the proper initial conditions given by the state of the line and the terminal apparatus at the start completely determines u_A, so that (14) gives the desired solution. This simple result could have been obtained in a perfectly elementary manner from the solution

$$u = F_1(x + y) + F_2(x - y)$$

or by the method of the Fourier integral.

As a very simple example, let us consider the case of a line initially free from current and potential to which a constant electromotive force E is suddenly applied through a coil of inductance L_0 and resistance R_0. Then we have

$$(21)\qquad x = 0, \quad L_0\frac{\partial I}{\partial t} + R_0 I + V = E,$$

or

$$L_0\frac{\partial u}{\partial y} + (R_0 - L_0 q)u - \sqrt{\frac{L}{K}}v = Ee^{qu}$$

and the solution is

$$I = 0, \quad x > t/\sqrt{KL};$$

$$(22)\qquad I = \frac{E}{R_0 + \sqrt{L/K}}\{1 - e^{-(R_0 + \sqrt{L/K})(t - x\sqrt{KL})/L_0}\} \quad (0 \leqq x \leqq t/\sqrt{KL}).$$

We thus see the discontinuity at the wave front, and we find that the line reacts on the terminal system like an extra resistance equal to the inductance of the length of line traversed by the wave in unit of time.

Let us now pass to the case of an actual line $p \neq 0$.

$$(23)\qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + p^2 u = 0,$$

$$(24)\qquad u = f(x), \quad \frac{\partial u}{\partial y} = h(x), \quad y = 0, \quad x \geqq 0.$$

The so-called Riemann function G is not now constant, but is such a solution as is equal to the constant unity along the characteristics PA, PB, and accordingly depends not only on the coordinates x, y, but on ξ, η, the coordinates of P. We will therefore write $G(x, y, \xi, \eta)$. If we should integrate our formula (11) around $PACBP$ we should now obtain the integral along the line AC, involving values of u which are unknown. To avoid this we integrate along the quadrilateral $PAOBP$, obtaining

$$PA, \quad \int_P^A Gdu - udG = (uG)_A - (uG)_P = u_A - u_P,$$

$$BP, \quad \int_B^P udG - Gdu = (uG)_B - (uG)_P = u_B - u_P,$$

$$AO, \quad \int_A^O \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)dy,$$

$$OB, \quad \int_O^B \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)dx = \int_O^B \left(Gh(x) - f(x)\frac{\partial G}{\partial y}\right)dx,$$

(24)

(25)
$$2u_P = u_A + u_B + \int_O^B \left(Gh(x) - f(x)\frac{\partial G}{\partial y}\right)dx + \int_O^A \left(u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x}\right)dy.$$

As before, we have to obtain the values of u, $\partial u/\partial x$ along OA. Proceeding as before, we apply our formula to the triangle OAC, but as we wish the Riemann function to be constant along AC we do not use the same G as before, but consider P moved to A, so that we put $G_1 = G(x, y; 0, Y)$. We thus obtain

(26)
$$u_A = u_C + \int_O^C \left(G_1 h(x) - f(x)\frac{\partial G_1}{\partial y}\right)dx - \int_O^A \left(G_1\frac{\partial u}{\partial x} - u\frac{\partial G_1}{\partial x}\right)dy.$$

We have now to put, according to Picard,

(27) $$G(x, y; \xi, \eta) = j\{p^2[(y-\eta)^2 - (x-\xi)^2]\},$$

where $j(x) = J_0(\sqrt{-x})$, J_0 representing the Bessel function of

order zero. We easily find the values

$$\frac{\partial G}{\partial x} = 2p^2(\xi - x)j'\{p^2[(y-\eta)^2-(x-\xi)^2]\},$$

$$\frac{\partial G}{\partial y} = 2p^2(y-\eta)j'\{p^2[(y-\eta)^2-(x-\xi)^2]\},$$

(28) OB, $G(x, 0; \xi, \eta) = j\{p^2[\eta^2-(x-\xi)^2]\}$,

$$\frac{\partial G}{\partial y} = -2p^2\eta j'\{p^2[\eta^2-(x-\xi)^2]\},$$

OC, $G_1(x, 0; 0, Y) = j\{p^2[Y^2-x^2)]\}$,

$$\frac{\partial G_1}{\partial y} = -2p^2 Y j'\{p^2[Y^2-x^2]\},$$

OA, $G(0, y; \xi, \eta) = j\{p^2[(y-\eta)^2-\xi^2]\}$,

$$\frac{\partial G}{\partial x} = 2p^2\xi j'\{p^2[(y-\eta)^2-\xi^2]\},$$

$$G_1(0, y; 0, Y) = j\{p^2[(y-Y)^2]\}, \quad \frac{\partial G_1}{\partial x} = 0.$$

Inserting these in (26)

$$(29)\quad u_A = u_C + \int_o^Y (h(x)j\{p^2[Y^2-x^2]\} + 2p^2 f(x) Y j'\{p^2[Y^2-x^2]\})dx$$

$$-\int_o^A j\{p^2[(y-Y)^2]\}\frac{\partial u}{\partial x}dy.$$

All the functions in the first integrand being known, the integral is a known function of its limit Y, so that (29) is an integral equation between u and $\partial u/\partial x$ of the form

$$(30)\qquad u(y) = \varphi(y) - \int_0^y K(x, y)F(x)dx,$$

where the kernel is symmetrical,

$$(31)\quad K(x, y) = j\{p^2[x-y]^2\}, \quad K(y, y) = 1, \quad F(y) = \left(\frac{\partial u}{\partial x}\right)_{x=0},$$

reducing to (16) when $p = 0$.

Differentiating successively by y, we obtain

$$\frac{\partial u}{\partial y} = \varphi'(y) - F(y) - \int_0^y \frac{\partial K}{\partial y} F(x)dx,$$

$$(32) \quad \frac{\partial^2 u}{\partial y^2} = \varphi''(y) - F'(y) - \left(\frac{\partial K}{\partial y}\right)_{x=y} F(y) - \int_0^y \frac{\partial^2 K}{\partial y^2} F(x)dx,$$

. . . . . .

From the differential equation (3) by integration we have

$$(33) \quad \frac{\partial v}{\partial y} - pv = F(y), \quad v = \int_0^y e^{p(y-x)} F(x)dx + v_0,$$

from which we may obtain the derivatives of v linearly in terms of those of F and an integral. Substituting all derivatives in the boundary condition (8) we have

$$A_0 F(y) + A_1 F'(y) + A_2 F''(y) + \cdots + \int_0^y \Phi(x, y) F(x)dx = \Psi(y),$$

which is a linear integro-differential equation of Volterra's form.* Solving this for $F(y)$ and inserting in (29) we obtain the desired solution from (25).

We thus see that the question of the reflection of waves at any terminal apparatus is contained in the values of u and $\partial u/\partial x$ along the line OA.

After the preceding paper was completed, I succeeded in seeing Goursat's paper in the *Annales de l'Université de Toulouse*, volume 6 (1904), page 117, and found that he had solved the case of u given along two lines making an acute angle, by means of a Volterra *integral* equation, which is a particular case of the above solution.†

CLARK UNIVERSITY, WORCESTER,
July 5, 1911.

* *Rendiconti dell' Acc. dei Lincei*, 1909, 1910.

† An abstract of the above paper was presented in the *Comptes Rendus*, Aug. 28, 1911.

NON-EUCLIDEAN GEOMETRY.

Bibliography of Non-Euclidean Geometry including the Theory of Parallels, the Foundations of Geometry, and Space of n Dimensions. By DUNCAN M. Y. SOMMERVILLE. London, Harrison and Sons, 1911. xii+404 pages.

ALMOST simultaneously with its quincentenary celebration, the University of St. Andrews has published this valuable compilation, the result of nine years of research on the part of one of its lecturers in mathematics. The title inevitably suggests similar lists by Halsted, Stäckel and Engel, Schlegel, and the lamented Bonola. To the work of all these bibliographic forebears Dr. Sommerville gives scrupulously exact reference. The bibliography of Halsted, "a model of its kind," includes nearly 200 titles of works relating to non-euclidean geometry and space of n dimensions, from about 1830 to 1879. Short notes appended to the chief works "form a valuable feature." Stäckel and Engel's bibliography on the theory of parallels is a chronological list of nearly 300 titles from 1482 to 1837 and, as Dr. Sommerville remarks, is almost complete. Schlegel's list of works on n dimensions accompanies a report on the subject, which supplies the place of Halsted's notes. It contains about 400 titles arranged alphabetically under the authors. The bibliography of Bonola is the most extensive which has recently appeared. It contains over 900 titles chronologically arranged from 1839 to 1902 with an index of authors. There is a classification under the headings Elementary geometry, Metrical and differential geometry, Group theory, Projective geometry, Finite distances, Vectorial methods, Mechanics and physics, General expositions, Philosophy and history, but there is no subject index.

Dr. Sommerville's idea at first was to prepare a continuation of Halsted's work but the growth of the subject rendered such diffuse treatment impossible and he was led to "produce as far as possible a complete repository of the titles of all works, from the earliest times up to the present, which deal with the extended conception of space, and to form a guide to the literature in an easily accessible form. It includes the theory of parallels, non-euclidean geometry, the foundations of geom-

etry and space of n dimensions. Works on the foundations of mathematics or arithmetic are excluded except in so far as they have an explicit reference to geometry."

The titles of which Dr. Sommerville takes account are over 4,000 in number. They may be roughly classified as follows: theory of parallels 700, non-euclidean geometry and the foundations of geometry 1,600, n dimensions 1,800. The approximate analysis of the nationalities of the authors (excluding mere reviewers and translators) gives: German, 460; French, 230; Italian, 220; British, 160; American, 90; Russian and Polish, 80; Dutch, 35; Hungarian, Bohemian, Croatian, etc., 35; Scandinavian, 35; no attempt is made to distinguish between German and Austrian or between Frenchman, Belgian, and Swiss. The total number of titles in the German language is 1,159; in French 884, Italian 848, English 723, while other languages represented by less than 100 titles are: Russian 99, Latin 80; Dutch 79, Magyar 37, Spanish and Portuguese 27, Danish and Norwegian 27, Polish 22, Swedish 11, Croatian 9, Greek 9, Czech 6, Arabic 4, Esperanto 4, Ruthenian 2, Roumanian 1, Sanskrit 1. The fact that no less than 1,470 titles are assigned to the decade 1901–1910 is an interesting indication of the trend of mathematical research at that time.

Riccardi's Bibliografia Euclidea was the model followed by Dr. Sommerville in the division of his work into three parts: I (pages 1–260) Chronological Catalogue, which covers the period from the time of Aristotle and Plato, who in the fourth century B.C. discussed the parallel postulate and the definitions of point and line, down to June, 1911, although no general attempt was made to obtain titles of later date than 1910; II (pages 251–316) Subject Catalogue in which the classification of the 1908 edition of the Index du répertoire bibliographique is used in amplified form; III (pages 317–398) Author's Index which, as far as possible, gives full names together with year of birth and, in the case of deceased authors, of death. Under each author's name is an abbreviated title and date of each of his writings, the details concerning which are to be found in I.* In the chronological catalogue the titles in each year are arranged alphabetically according to the authors. The various

* The seven persons who have the greatest number of titles (including biographical sketches and reviews) to their credit in Dr. Sommerville's list are, in order: Halsted (77), Mansion, Schoute, Segre, Poincaré, Klein, Bianchi.

editions, translations and reviews * of a work as well as its classification (II) and price, when a separate work, are here found. References to bibliographical sources have only been given in cases of discrepancies among the authorities. A very few titles are marked with an asterisk (*) as having been personally verified by the author. This fact naturally inspires the fear that the subject classification will require considerable emendation to make it an accurate epitome of the work done in the various fields. On pages 399–404 are given "additions" and "corrections."

It may be asked, to what extent has the author succeeded in making a complete work within the prescribed limits? On looking through the list of his sources of information we find beside the bibliographies mentioned above, the *Jahrbuch*; *Revue Semestrielle*; International Catalogue (A); Royal Society Catalogue, Authors, volumes 1–12, Subject, volume 1;† Poggendorff; and Wölffing's Mathematischer Bücherschatz. It is with some surprise that we learn that no use was made of the Royal Society Catalogue, Subject Index, volume 2, "Mechanics," which would have supplied at least four additional titles of value, or of the International Catalogue, (B) Mechanics, (C) Physics, or of the 1,900 slips of the Répertoire bibliographique des Sciences mathématiques. Nor do we anywhere find references to Felix Müller's excellent Führer durch die mathematische Literatur with its well arranged geometry bibliographies which cover many pages. We are therefore prepared for numerous omissions (the more, when we recall the well known imperfections of most of the above mentioned works) and it is indeed probable that they number many hundreds, even though Dr. Sommerville has already digested a great mass of material. A very brief examination of his book suggested a score of additions:‡ D. Lardner, Elements of Euclid, third edition, London, 1832; sixth edition, 1838; eighth, 1843; ninth, 1846; tenth, 1849; W. C. Hume, A treatise on the theory of parallel lines, Dublin, 1853, pages viii + 40 and 2 plates; E. Padova [extension to n-space of two theorems by Neumann on potential], *Giornale di Matematiche*, Napoli, volume 8 (1870), pages 296–301; J. König [Traité ana-

* It is to be regretted that the names in connection with signed reviews are rarely given.

† Dr. Sommerville has, apparently, overlooked the Appendix of this volume.

‡ It is to be understood, of course, that not all the following titles would be noticed except in a work aiming to be as complete as the one under review.

lytique des hypothèses de la géométrie], *Értekezések a Mathematikai Osztály Köréből. Kiadja a Magyar Tudományos Akadémia*, Budapest, volume 6 (1872), pages 94–98; A. Tonelli [Potential function in n-space], *Göttinger Nachrichten*, 1875, pages 521–552; É. Lemmi, "Sur les cas d'exception au théorème des forces vives . . .," *Journal de Mathématiques pures et appliquées*, Paris, volume 2 (1876), pages 233–239; Réthy [Trigonometry in non-euclidean space], *A Magyar Tudományos Akadémia Értesítoje*, Budapest, volume 25 (1876); F. W. E. A. Kellner [Considerations on the theory of parallels as basis of geometry], Amsterdam, 1879, 81 pages; O. Simony, "Ueber eine Reihe neuer mathematischer Erfahrungssätze," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften*, Wien, volume 87 (1883), pages 556–587; volume 88 (1884), pages 939–974; F. M. Souvorof [Sur la formule générale pour la distance de deux éléments dans un système projectif à une dimension], *Proceedings of the Physico-Mathematical Section of the Naturalists' Society of the Imperial University of Kazan*,* volume I (1883) pages 9–11; J. C. Medeiros [Complément à la théorie des parallèles d'Euclide], *O Instituto, jornal scientifico e litteraris*, Coimbra, volume 30 (1883), pages 541–553; L. Kronecker [Clausius's coordinates in n-space], *Berliner Sitzungsberichte*, 1891, pages 881–890; Klein, "Vorlesungen über projektive Geometrie" (mimeographed), Cambridge, Mass., 1893;† A. Vasiliev, N. I. Lobachevsky, *El Progreso matematico*, volume 3 (1893), pages 137–139; J. N. Lyle, various papers in the *American Mathematical Monthly*, volumes 1–3 (1894–1896); in volume 3 (1896) of the *Monthly* there was also a paper by J. H. Drummond, in volume 11 (1904) one by E. B. Wilson, in volume 16 (1909) one by Wilczynski. No reference is given to Klein's "Elementarmathematik vom höheren Standpunkte aus," Teil II: Geometrie, Leipzig, 1909, or to an 8 page (large quarto size) sketch (in French) of Lobachevsky's life and works published, according to a pencilled note on my copy, at Kazan in 1886. The greater part (pages 581–593‡) of Note II ("Sur la géométrie non-euclidienne") in the seventh edition of Rouché and Comberousse's Géométrie, volume 2, Paris, 1900,

* Translation of Russian title.

† The edition of about 25 copies was published by some graduate students of Harvard University.

‡ E. Lebon in the Poincaré volume of the Savants du Jour series makes a slip in connection with this entry.

should have been credited to Poincaré. "Rouché and Comberousse" is incorrectly listed under the date 1891, of its sixth edition; in the fifth edition of 1883 the notes treat of the same subjects as in the later editions. Miss Scott's articles in volume 1 (1900) of the *Mathematical Gazette* should surely be mentioned in connection with von Staudt's Geometrie der Lage, and if two English editions of Newcomb's Popular Astronomy are to be listed, reference should also be give to the 8 American, the 4 German, the Russian, and the Norwegian editions. The entry after "[Crelle, A. L.]" under 1835 should be put in the list for 1834. To the title under Eisenmenger's name on page 52 may be added: pp. 14+1 pl.

Incomplete as the work before us therefore appears, the mathematician must ever be grateful to Dr. Sommerville for this new, carefully prepared and admirably arranged aid to orientation in a vast field of mathematical science.

The volume is well printed by His Majesty's Stationers and the general get up (including the paper binding) is the same as for the International Catalogue.

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NOTES.

THE thirtieth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on April 5–6.

THE opening (January) number of volume 13 of the *Transactions of the American Mathematical Society* contains the following papers: "Ueber eine idealtheoretische Funktion," by E. LANDAU; "Theorems of oscillation for two linear differential equations of the second order with two parameters," by R. G. D. RICHARDSON; "The absolute minimum of a definite integral in a special field," by E. J. MILES; "An existence theorem for a problem of the calculus of variations in space," by E. G. BILL; "Linear algebras," by L. E. DICKSON; "A note concerning Veblen's axioms for geometry," by R. L. MOORE; "Natural families of curves in a general curved space of n dimensions," by JOSEPH LIPKE; "A class of periodic orbits of superior planets," by F. R. MOULTON; "Harmonic functions and Green's integral," by O. D. KELLOGG.

THE opening (January) number of volume 34 of the *American Journal of Mathematics* contains: "Primitive groups with a determination of the primitive groups of degree 20," by ELIZABETH R. BENNETT; "On arithmetical theory of pencils of binary quadratic forms," by D. N. LEHMER; "On the principle of duality in the geometry of the sphere," by ARTHUR RANUM; "On the fundamental postulate of tamisage," by R. K. MORLEY; "Derivative domains of rationality," by S. Epsteen; "Critical revision of de Haan's Tables of definite integrals," by E. W. SHELDON. The frontispiece of the volume is a portrait of Maclaurin.

THE December number (volume 13, number 2) of the *Annals of Mathematics* contains: "Points of indeterminate slope on the discriminant locus of an ordinary differential equation," by W. R. LONGLEY; "Boundary problems and Green's functions for linear differential and difference equations," by MAXIME BÔCHER; "Conjugate directions on a hypersurface in a space of four dimensions and some allied curves," by C. L. E. MOORE.

AT the meeting of the London mathematical society held on December 14 the following papers were read: By J. W. NICHOLSON, "The pressure of radiation on a cylindrical obstacle"; by H. HILTON, "Hermitian invariants of a canonical substitution"; by E. W. HOBSON, "The fundamental lemma of the calculus of variations and some related theorems"; by W. BURNSIDE, "The outer isomorphisms of a group"; by E. B. STOUFFER, "Invariants of linear differential equations"; by J. C. FIELDS, "A method of proving certain theorems relating to adjointness."

AT the meeting of the Edinburgh Mathematical Society on January 12 the following papers were read: By J. A. DONALDSON, "An infinite series of triangles and conics with a common pole and polar"; by R. F. DAVIS, "The condition that the family of surfaces $\phi(x, y, z, \lambda) \doteq 0$ may be one of an orthogonal triad"; by F. E. EDWARDS, "Graphical trisection of circular arc."

THE annual meeting of the Mathematical association was held in London, January 10. The following papers were read. Presidential address, by E. W. HOBSON, "On the work of the international commission on mathematical teaching," by C.

GODFREY; "On some unrealized possibilities in mathematical education," by G. ST. L. CARSON; "A plea for the earlier introduction of the calculus, by C. V. DURELL.

THE United States Bureau of Education has recently issued Bulletins 13 and 16 for 1911, the former containing the report of the American Committees I and II, on mathematics in the elementary schools of the United States, the latter the report of Committees III and IV, on mathematics in the public and private secondary schools. These reports are prepared under the direction of the American Commissioners of the International commission on the teaching of mathematics. They may be secured gratis by addressing the United States Commissioner of Education at Washington.

AT the University of Göttingen mathematical statistics has recently been added to the list of possible subjects from which the major and minors can be chosen for examination for the doctorate. Candidates in applied mathematics must henceforth be prepared to be examined in the mathematics of insurance.

DR. G. SCHIRMER, a Chicago physician, has established at the University of Erlangen, a prize fund, to be known as the Helene Ottilie Schirmer foundation, in honor of his late wife. The income of about 600 marks is to be given to the author of the most meritorious thesis prepared at the university during a period of two years preceding each award; the subjects are to be in mathematics or physics in odd years, and in medicine in even years. The first award has recently been made to Dr. R. BALDUS for his dissertation on certain line congruences.

AT the Collège de France the chair of mathematics has been replaced by two, one in analysis and the other in geometry. Professor J. HADAMARD has been appointed professor of analysis and Dr. M. D'OCAGNE has been appointed professor of geometry. Dr. VESSIOT has charge of the mathematical examinations for entrance and advancement. The lectureship under the Peccot foundation has been divided; during the first half year it will be filled by Dr. E. CHAZY and during the second half year by Dr. L. CHATENET. The Peccot prize for the year 1911 was awarded to Dr. M. JANET.

THE Belgian academy has crowned a memoir on analysis by S. BERNSTEIN and awarded the F. Deruyts prize for geometry to Dr. E. FAIRON.

THE London mathematical society has awarded its De Morgan medal to Professor HORACE LAMB.

THE fiftieth scientific anniversary of Professor GASTON DARBOUX was celebrated at the Sorbonne on January 21.

PROFESSOR V. VOLTERRA, of the University of Rome, is giving a series of lectures at the University of Paris on the extension of the theory of functions, the integro-differential equations and integral equations, with applications. The first lecture was given January 22.

PROFESSOR A. VOSS, of the University of Munich, has been decorated with the Bavarian order of Maximilian.

PROFESSOR F. ENRIQUES, of the University of Bologna, has received the honorary degree of doctor of laws from the University of St. Andrews.

PROFESSOR J. J. THOMSON, of the University of Cambridge, has been elected foreign member of the royal academy of sciences of Naples.

PROFESSOR G. LAURICELLA, who had recently accepted a professorship of mathematics at the University of Rome, has at his own request resumed his former duties at the University of Catania.

PROFESSOR P. ANDOYER has been elected president of the Société mathématique du France.

DR. F. NUSL, of the Bohemian technical school at Prague, has been promoted to a full professorship of mathematics.

DR. R. BALDUS has been appointed docent in mathematics at the University of Erlangen.

DR. K. KNOPP has been appointed docent in mathematics at the University of Berlin.

PROFESSOR G. FABER, of the technical school at Stuttgart, has accepted a professorship of mathematics at the University of Königsberg.

DR. K. KOMMERELL has been appointed docent in mathematics at the technical school at Stuttgart.

DR. E. KRUPPA has been appointed docent in geometry at the University of Czernowitz.

PROFESSOR H. HENRIQUES, of the technical school at Stockholm, has been promoted to a full professorship of descriptive geometry.

PROFESSOR K. WIEGHARDT, of the technical school at Hanover, has accepted a professorship of mechanics at the technical school of Vienna.

DR. F. NOETHER has been appointed docent in mathematics at the technical school of Karlsruhe.

DR. H. STRÖLE has been appointed docent in mathematics at the University of Neuchâtel.

DR. A. TIMPE, of the technical school at Aachen, has been appointed docent in mechanics at the University of Münster.

PROFESSOR P. BURGATTI, of the University of Bologna, has been promoted to a full professorship of theoretical mechanics.

DR. Z. GIAMBELLI has been appointed associate professor of algebra at the University of Cagliari.

PROFESSOR E. C. COLPITTS, of the College of Emporia, Kansas, has been appointed professor of mathematics at the State College of Washington, at Pullman, Washington.

DR. A. B. FRIZELL, of the University of Kansas, has been appointed professor of mathematics in McPherson College, McPherson, Kansas.

THE death is announced of Professor GEORGE CHRYSTAL, of the University of Edinburgh, at the age of 60 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDING (E.). Sechsstellige Tafeln der Besselschen Funktionen imaginären Arguments. Leipzig, Engelmann, 1911. 8vo. 4+72 pp. Cloth. M. 8.80

ARTUSBRUNNEN (C.). Die Gleichung für 3 Potenzen gleicher Ordnung mit der Einschränkung des Fermat. Ein Beweis für die Grundregel der Mathematik. (Das Problem des Pythagoras.) 2te vermehrte und neudurchgesehene Ausgabe. Berlin, Buschhardt, 1911. 8vo. 4 pp. M. 2.00

AUSSANT-CARÀ (P.). Sulla discussione dei problemi riducibili al secondo grado. Con varie applicazioni. Livorno, Giusti, 1911. 8vo. 58 pp. L. 1.50

BERNADI (G.). Tavole contenenti i doppi i quadrati, i tripli dei quadrati ed i cubi dei numeri interi da 1 a 1000, ad uso degli istituti medi e superiori d'istruzione. 2a edizione rifatta. Bologna, Beltrami, 1911. 16mo. L. 1.50

BERZOLARI (L.). Geometria analitica, I: Il metodo delle coordinate. Milano, Hoepli, 1911. 16mo. 16+409 pp. L. 3.00

BÖHM (F.). Über die Transformationen von homogenen bilinearen Differentialausdrücken. (Habilitationsschrift.) München, 1911.

DEDEKIND (R.). Was sind und was sollen die Zahlen. 3te unveränderte Auflage. Braunschweig, Vieweg, 1911. 8vo. 21+58 pp. M. 1.80

DINGLER (H.). Über die Bedeutung der Burali-Fortischen Antinomie für die Wohlordnungssätze der Mengenlehre. München, Ackermann, 1911.

FRECHET (M.). See HEYWOOD (H. B.).

GOLDMANN (H.). Der russische Rechenapparat, seine rationelle Anwendung in den Schulen und im praktischen Leben, nebst einigen Untersuchungen über den bildenden Wert der Arithmetik. Wien, Pichler, 1911. 8vo. 3+44 pp. M. 0.85

HEYWOOD (H. B.) et FRECHET (M.). L'équation de Fredholm et ses applications à la physique mathématique, avec une préface de M. J. Hadamard. Paris, Hermann, 1911. 8vo. 165 pp. Fr. 5.00

JACKSON (D.). Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung. Göttingen, 1911 (Gekrönte Preisschrift.) 8vo. 98 pp.

KISELJAK (M.). Beiträge zur Theorie der vollkommenen Zahlen. (Croatian.) (Progr.) Agram, 1911.

LALESCO (T.). Introduction à la théorie des équations intégrales, avec une préface de E. Picard. Paris, Hermann, 1911. 8vo. 152 pp. Fr. 4.00

LESSER (O.). Die Infinitesimalrechnung im Unterricht der Prima. 2te Auflage. Berlin, Salle, 1911. M. 1.80

MEYER (W. F.). Über die Theorie benachbarter Geraden und ein verallgemeinerter Krümmungsbegriff. Eine Ergänzung zu den Lehrbüchern über Differentialgeometrie. Leipzig, Teubner, 1911. 8vo. 18+152 pp. M. 9.00

MÜLLER. Die Grundbegriffe der Differential- und Integralrechnung und ihre Anwendung auf die Elementarmathematik. Eine Darstellung in Aufgaben. Metz, Scriba, 1911. M. 0.70

PRÜSMANN (R.). Neue Auflösungen der Gleichung 5ten Grades auf Grund linearer Gravitationen. Berlin, Weidmann, 1911. M. 2.40

PRYM (F.) und ROST (G.). Theorie der Prym'schen Funktionen erster Ordnung im Anschluss an die Schöpfungen Riemann's. Leipzig, Teubner, 1911. 4to. 11+300 pp.

ROST (G.). See PRYM (F.).

SHINE (MRS. M. G.). Little journeys into the invisible: a woman's actual experience in the fourth dimension. Richmond, Va., Allshine, 1911. 16mo. 71 pp. $1.00

TIMERDING (H. E.). Die Infinitesimalrechnung auf der Schule. Leipzig, Teubner, 1911. M. 0.80

TURRIÈRE (E.). Sur les congruences de normales qui appartiennent à un complexe donné. (Thèse.) Toulouse, Privat, 1911. 4to. 87 pp.

VÖRÖS (C.). Analitika geometrio absoluta. Vol. II: La spaco Bolyaia. Budapest, Kókai, 1912. 8vo. 199 pp. M. 9.60

WIELEITNER (H.). Der Begriff der Zahl. (Mathematische Bibliothek herausgegeben von W. Lietzmann und A. Witting. Nr. 2.) Leipzig, Teubner, 1911. M. 0.80

WIENER (F. W.). Elementare Beiträge zur neueren Funktionentheorie. (Diss.) Göttingen, 1911.

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BALDASSERONI (G.). Aritmetica e geometria, con brevi nozioni di computisteria pratica, per la quinta classe elementare con moltissimi esercizi e problemi. Quinta edizione, accuratemente riveduta. Firenze, Bemporad, 1911. 16mo. 159 pp. L. 0.60

——. L'aritmetica e la geometria nella quarta classe elementare. Firenze, Bemporad, 1911. 16mo. 76 pp. L. 0.40

BETTINI (B.) e CIAMBERLINI (C.). Elementi di algebra pratica per le scuole d'arti e mestieri e per i corsi inferiori delle scuole industriali. Livorno, Giusti, 1911. 16mo. 194 pp. L. 1.50

CAMMAN (P.) et GRIGNON (A.). Algèbre. Classes de troisième B, seconde et première C et D. Paris, Gigord, 1912. 16mo. 7+279 pp.

CATANIA (S.). Elementi di aritmetica ed algebra per le scuole secondarie inferiori. 2a edizione rifatta. Catania, Gianotta, 1911. 16mo. 320 pp. L. 2.50

CIAMBERLINI (C.). Aritmetica e geometria, con molti esercizi e problemi di calcolo mentale e scritto, e nozioni di computisteria pratica, ad uso degli alunni della 5a classe elementare. Firenze, Bemporad, 1911. 16mo. 128 pp. L. 0.60

——. Nozioni di aritmetica e geometria con esercizi di calcolo mentale e scritto e nozioni di computisteria pratica, per la 5a e 6a classe elementare. Nuova edizione, completemente rifusa. 2 volumi. Firenze, Bemporad, 1911. 16mo. 175+173 pp. L. 1.55

——. See BETTINI (B.).

COSTA REGHINI (A.). Appunti di trigonometria piana e sferica. Ancona, Fogola, 1912. 16mo. 96 pp. L. 1.00

DURELL (F.). Durell's school algebra. New York, Merrill, 1911. 507 pp. $1.10

ENRIQUES (F.) e AMALDI (U.). Corso completo di geometria ad uso delle scuole medie. Elementi di geometria per le scuole secondarie superiori. 4a edizione. Bologna, Zanichelli, 1911. 16mo. L. 4.50

ERNST (E.). Mathematische Unterhaltungen und Spielereien für Jung und Alt. 1tes Heft. Ravensburg, Maier, 1911. 8vo. 100 pp. M. 1.00

EVANS (G. W.). The teaching of high school mathematics. (Riverside educational mongraphs.) Boston, Houghton Mifflin, 1911. 16mo. 9+93 pp. $0.35

FEUKNER (H.). Lehrbuch der Geometrie. Ausgabe A: für Gymnasien, Realgymnasien und Ober-Realschulen. Zweiter Teil: Raumgeometrie. 4te Auflage. Berlin, Salle, 1911. M. 1.80

——. Lehrbuch der Geometrie. Ausgabe A. Dritter Teil: Ebene und sphärische Trigonometrie. 2te Auflage. Berlin, Salle, 1911. M. 2.00

FORNEL DE LA LAURENCE (de). Introduction aux mathématiques, suivie d'une étude sur les logarithmes. Paris, Delagrave, 1911. 18mo. 93 pp.

FORTIN (C.). Cours de trigonométrie conforme aux programmes de 1905. Paris, Desclée, 1911. 16mo. 22+249 pp.

GRIGNON (A.). See CAMMAN (P.).

GUIMARES (R.). Les mathématiques en Portugal. Appendice II. Coïmbre, Imprimerie de l'Université. 8vo. 107 pp.

HECHT (C.) und KLÄRNER (K.). Mathematisches Lehrbuch für Knabenschulen. Raumlehre. (Planimetrie, Trigonometrie und Stereometrie.) Nach den Bestimmungen über die Neuordnung des Mittelschulwesens in Preussen vom März 2, 1910, bearbeitet von K. Klärner. Bielefeld, Velhagen und Klasing, 1911. 8vo. 6+230 pp. Cloth. M. 2.00

HELLY (E.). Lösungen der Aufgaben in Suppantschitsch Lehrbuch der Geometrie für die 4ten und 5ten Klasse der Gymnasien und Realgymnasien. Wien, Tempsky, 1911. 8vo. 92 pp. M. 1.60

HOČEVAR (F.). Lehr- und Übungsbuch der Geometrie für Realschulen. Mittelstufe (4te und 5te Klasse). 3te Auflage. Wien, Tempsky, 1911. 8vo. 171 pp. Cloth. M. 2.60

JAHNKE (E.). Die Mathematik an Hochschulen für besondere Fachgebiete. (4ter Band, 7tes Heft der Abhandlungen über den mathematischen Unterricht in Deutschland.) Leipzig, Teubner, 1911. 8vo. 6+55 pp. M. 1.80

KLÄRNER (K.). See HECHT (C.).

LAFFIN (U. S.). Mathematical construction. Chicago, Flanagan, 1911. 129 pp. $0.50

LIETZMANN (W.). See SCHUSTER (M.).

LINDT (R.). Misstände im Unterricht und im Prüfungswesen der Hochschulen und ihre Beseitigung. Charlottenburg, Heydenreich, 1911. M. 0.65

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